

Modeling of CPDOs
Identifying Optimal and Implied Leverage

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Abstract

Considering the recent subprime crisis, the demand for collateralized products based on Credit Default Swap (CDS) exposures combined with security features is increasing. Constant Proportion Collateralized Debt Obligations (CPDOs) are a mixture of Collateralized Debt Obligations (CDOs) and CPPIs with inverse mechanism. This new asset targets to meet the upcoming investors' demand in the credit derivatives market, but quantitative approaches for pricing except for simulation algorithms do not exist yet up to the author's knowledge. This article aims to shed a light on CPDOs specific structural enhancements and mechanisms. The author quantifies inherent risks and provides a closed-form pricing formula.

Economic Rational

A Constant Proportion Collateralized Debt Obligation (CPDO) can be interpreted as a hybrid credit derivative. In fact the inverse mechanism of a Constant Portfolio Insurance (CPPI) strategy is applied to a Collateralized Debt Obligation (CDO). This asset class came up for the first time in 2006 and encountered quite a success. Recall that a standard synthetic CDO, i.e. a CDO based on a CDS portfolio, is not capable to redeem the noteholders before maturity. That means if an investor or the issuing bank decided to call a synthetic CDO deal before maturity for any reasons (issuing bank wants to prevent their clients from upcoming market crisis; the investor got stuck in a liquidity gap and has no more confidence in the pricing provided by the originator), there exists a risk that the trancheholder does not get the notional of his tranche redeemed. Now that is not a big revelation as this characteristic is inherent to CDO structures: At issuance the notional of the underlying CDO portfolio is inferior to the notional of the issued tranches. This gap is called the *Shortfall*. This shortfall has to pay for the issuing costs and guarantees the margin for the issuer. Consequently the shortfall is supposed to be reduced to 0 by future CDS premium payments until maturity. Obviously, inconvenient market conditions as well as defaults of credit entities transform the supposed shortfall reduction into a risky bet.

Now a CPDO structure aims at accelerating the reduction of the shortfall over the CPDO lifetime via an inverse CPPI mechanism: A CPDO relies on an underlying

CDS portfolio *and* a risk-free Cash Account (cf. figure 1). As CDS spreads widen (and thus the CDS portfolio becomes riskier) the exposure to the CDS market (called the Risky Exposure) increases. In fact the selling of additional protection is financed by the cash account which consequently decreases. In order to keep proportions between Risky Exposure and Cash Account constant, the originating bank fills up the Cash Account. The concept might sound paradoxical: why should one increase his exposure on the CDS market while the latter becomes riskier? In literature, we often find the following illustrating example:

One can draw a parallel with a simple coin-toss game. The outcome of the toss is either "heads" or "tails". Heads results in a 100% return and tails a 100% loss. The player has an initial stake of 1000. If he bets 10 from the initial stake on "heads", there are two possible outcomes:

- "Heads": the player's stake rises to 1010, so at the next round he will bet only 9.90.
- "Tails": he now has only 990, so at the next round he will bet 10.10.

Such a strategy is based on the notion that if "heads" has appeared more frequently than expected, it is less likely to continue appearing, and similarly for "tails". In other words, the strategy is based on the concept of mean reversion.

Now back to the CPDO mechanism: If the shortfall is reduced to 0 before maturity, the so-called *Cash-In* event occurs: The CPDO structure is unwound and the proceeds are invested in the Cash Account until maturity. This guarantees redemption to the trancheholder.

If the shortfall exceeds a certain threshold, there is a *Cash-Out* event: The CPDO structure is unwound and the proceeds are transferred to the investors. This event can be interpreted as a "Stop-Loss" strategy.

It becomes obvious that CPDO deals -despite of some inherent risks we will focus later on- meet investors' needs particularly after the recent subprime crisis.

1 Preliminaries

In the following we present current notions and a comprehensive scheme which are fundamental for comprehension of this article:

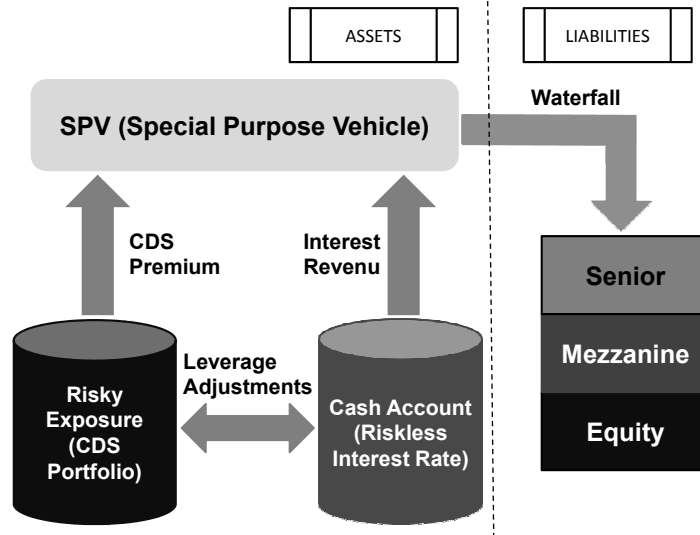


Figure 1: A classic CPDO scheme.

1.1 Current Notions

Risky Exposure (RE): The Risky Exposure represents the total notional of the underlying CDS portfolio.

CDS Mark-to-Market (MtM) value: Difference between contracted spread and current market spread.

Net Asset Value (NAV): The Net Asset Value represents the total value of the Assets, i.e. the Cash Account + the MtM value of the Risky Exposure.

Target Bond Price (TBP): The TBP represents the Present Value of the Liabilities (i.e. coupons to pay + redemption)

Shortfall: $TBP - NAV$. The shortfall measures the "financing gap" between assets and liabilities and thus dictates the value of protection sold.

Cash-In Event: If $NAV = TBP$ before maturity, the CPDO structure is unwound and the proceeds are invested in the Cash Account until maturity. This guarantees redemption to the trancheholder (hence the Risky exposure is reduced to 0).

Cash-Out Event: If $NAV < \text{Threshold}$ before maturity, the CPDO structure is unwound and the proceeds are restituted to the investors. This event can be interpreted as a "Stop-Loss" strategy.

1.2 The Credit CPPI Strategy

The credit CPPI strategy aims at maximizing returns while providing full/partial protection. The rules ensure a minimum acceptable return ("bond floor"):

Assume a CDS portfolio linked to a subportfolio of risk-free bonds. As CDS spreads widen (and hence risk and the portfolio value increases), the notional of the CDS portfolio is reduced and the resulting proceeds are invested in the risk-free bond (and vice versa). However if the value of the risky exposure falls below the bond floor, the RE gets liquidated and the proceeds are shifted towards the riskless subportfolio in order to achieve a minimum (risk-free) return.

Inconveniences: Once the portfolio value has fallen below the bond floor, the residual proceeds are invested in the Cash Account. As a result one has to ride the Libor/Euribor curve until the "virtual" portfolio value is superior to the bond floor. This leads to a waste of time for the investor as he got stuck with the riskless interest rate and is cannot take advantage of the increasing CDS spread rates. This type of risk is widely known as *Gap Risk*.

cf. Exhibit 1

1.3 The CPDO Strategy

The CPDO mechanism is the opposite of the Credit CPPI's. As CDS spreads widen and the value of the risky exposure decreases, the investor aims to take profit of the larger spreads and hence increases the notional of the risky exposure. Thus this strategy lacks principal protection (Cash-out barrier provides only partial protection). It does not maximize returns either as they are capped by the Cash-In event. The advantage consists in the capacity to recover from negative performance

as the investor is not penalized by Gap Risk. Of course this favors the principal aim of reducing the shortfall as quickly as possible until maturity.

It becomes obvious that the principal driver of the CPDO's performance consists in the NAV which itself is influenced by different factors, i.e. defaults, spread variation, interest rate variation or bid/offer gap variations (the latter three to an inferior extent).

Income Effect vs MtM Effect

We have explained so far that the shortfall is supposed to be reduced by future high spread revenues. An indicator for the optimal Risky Exposure is the so-called *Target Notional* which is defined as follows:

$$\text{Target Notional} = \frac{\text{Shortfall}}{\text{Risky Income}} \times \text{Gearing Factor}$$

The Gearing Factor can be interpreted as the "speed" of shortfall reduction as it exaggerates the real shortfall (hence more protection will be sold) in order to be on the safe side.

However in most practitioner-oriented literature, the Target Notional is seen as an optimal leverage which is false by definition of the leverage. In addition this assumption would also corrupt modeling. We treat this problem extensively in the following section.

Now imagine the following scenario: When spreads widen and the CPDO structure recontracts at the new rate, the risky income increases. This is called the *Income Effect*. However the NAV diminishes as the MtM value of the risky portfolio turns negative. Recall that the latter is defined as the difference between contracted spread minus market spread. Hence the shortfall increases. The question now is: which effect is dominant on the evolution of the Target Notional ? Obviously this depends on the remaining time interval until the structure recontracts at the new rate and hence eliminates the MtM variation of the risky portfolio.

cf. Exhibit 2. 3

1.4 The CPDO Mechanism

In order to illustrate the exact behavior of a CPDO structure, we present a comprehensive basic CPDO algorithm as shown in UBS [2007]. We define the following notations:

$S(t)$	Spot Spread Process
$S_c(t)$	Contracted Spread Rate
n	Number of time steps
$N_t^{Tranche}$	Notional of the issued tranche at time t
$C(\cdot)$	Quarterly coupon paid on the tranche (in %)
$B(t, T_j)$	Zero coupon bond with tenor $[t; T_j]$
λ	Default intensity
$L(t)$	Leverage Rate at t
R	Constant Recovery Rate (in %)

Step 1: Determining the Present Value of Liabilities and Assets

For $t < T_1, \dots < T_n$, the present value of the structure's liabilities is calculated as follows:

$$PV_{Liab} = \sum_{j=1}^n B(t, T_j) \times C(T_j) N_{T_j}^{Tranche} + B(t, T_n) \times (C(T_n) N_{T_n}^{Tranche} + N_{T_n}^{Tranche}) \quad (1)$$

In the same manner one can write:

$$PV_{Assets} = S_c(t) \times N_t^{Tranche} \times \sum_{j=1}^n B(t, T_j) e^{-\int_t^{T_j} \lambda(u) du} \quad (2)$$

$$= S_c(t) \times N_t^{Tranche} \times \sum_{j=1}^n B(t, T_j) e^{-\left(\frac{S(t)(T_j-t)}{1-R}\right)} \quad (3)$$

where $e^{-\left(\frac{S(t)(T_j-t)}{1-R}\right)}$ represents the market-implied equivalent of the risky discount rate $e^{-\int_t^{T_j} \lambda(u) du}$.

The multiplication with the market-implied risky discount rate is necessary as the calculation above is based on the contracted spread rate which does not reflect the market risk of the spot spread rate.

Proof. Consider a one-period CDS rate. The premium payments equal the protec-

tion payments and one can write

$$S(T_1)e^{-\int_0^{T_1} \lambda(u)du} = (1 - R)B(t, T_1) \quad (4)$$

$$\ln \left[\frac{S(T_1)}{(1 - R)} \right] = - \int_0^{T_1} \lambda(u)du \quad (5)$$

$$\ln \left[\frac{(1 - R)}{S(T_1)} \right] = \int_0^{T_1} \lambda(u)du \quad (6)$$

$$\frac{S(T_1)}{(1 - R)} = \lambda(T_1) \quad (7)$$

□

Step 2: Modeling the NAV

At issuance the MtM value of the Risky Portfolio equals zero. Hence

$$RE(0) = L(0) \times N_{PF}(0) \quad (8)$$

The MtM value is given by

$$MtM(t) = (S_c(t) - S(t)) \times RE(t) \times \sum_j B(t, T_j) e^{-\left(\frac{S(t)(T_j - t)}{1 - R}\right)} \quad (9)$$

One can deduct that if spread rates tighten, the MtM value of the risky exposure becomes positive. The MtM itself is realized by rebalancing between the CDS portfolio and the Cash Account as well as on a CDS Index roll date. The mechanism behind can be described as follows:

- When spread rates tighten, the structure buys back protection (i.e. sells CDS) which diminishes the Risky Exposure
- The widening of CDS spreads initializes the selling of additional protection which increases the notional of the risky exposure
- The Roll Date of A CDS Index leads to the recontracting of concerned CDS rates and hence the contracted spread rates widen/tighten. As a consequence the MtM value is set to zero and the Risky Exposure remains unchanged.

The second constituent of the NAV besides the Risky Exposure is the Cash Account.

The latter is defined as follows:

$$Account(t) = Account(t - 1)(1 + Libor(t)) + S_c(t) \times RE(t) - Loss(t) \quad (10)$$

$$- Fees(t) - C(t)N_t^{Tranche} \quad (11)$$

As stated previously

$$NAV(t) = MtM(t) + Account(t) \quad (12)$$

This leads to the following conclusions:

- If the NAV exceeds the present value of the structure's liabilities, the Cash-In event occurs. Consequently the risky exposure is reduced to zero.
- If the NAV falls beyond a pre-determined threshold, the Cash-Out event occurs and the structure is unwound.
- If NAV is inferior to the present value of liabilities, the algorithm continues with Step 3.

Step 3: Leverage Adjustments

One has to differentiate between *Optimal* and *Implied Leverage*.

Definition 1.1. "Optimal Leverage":

$$L_{opt}(t) = \frac{PV_{Liab}(t) - NAV(t)}{N_t^{Notes}} \times G \quad (13)$$

To the contrary to some literature, we define the optimal leverage in relation to the notional of the issued notes and not based on the present value of the structure's assets. Leverage is a measure of to what extent variations concerning the asset side are amplified and impact the liabilities. Thus leverage should always be defined as a fraction of liabilities and assets. The optimal leverage defined above allows for optimal allocation of the risky exposure in order to regain the shortfall.

Definition 1.2. "Implied Leverage:"

$$L_{imp}(t) = \frac{RE(t)}{N_t^{Notes}} \quad (14)$$

The implied leverage represents the "real" leverage. We have already described various factors which influence the value of the risky portfolio. Consequently one has to adjust the leverage in order to re-allocate the Risky Exposure.

If the implied leverage quotes outside a pre-defined interval, the Risky Exposure is adjusted in order to meet the CPDO structure's optimal leverage.

Consider an interval $I = [(1 - \epsilon)L_{opt}; (1 + \epsilon)L_{opt}]$. If $L_{imp} \notin I$, the CPDO manager

launches a rebalancing measure so that $L_{imp} = L_{opt}$. Hence the overall leverage fulfills the following requirements:

$$L(t) = L_{opt}(t) \times 1_{\{L_{imp} \notin I\}} + L_{imp}(t) 1_{\{L_{imp} \in I\}} \quad (15)$$

Consequently the post-rebalancing Risky Exposure can be written as follows:

$$RE^{post}(t) = L(t) \times N^{Notes}(t) \quad (16)$$

If

- $L(t) > L(t - 1) \Rightarrow$ additional protection is sold in order to increase the Risky Exposure
- $L(t) < L(t - 1) \Rightarrow$ protection is bought back in order to reduce the CDS portfolio

The re-balancing also engenders a new contracted CDS Spread:

$$S_c^{post}(t) = w(t)S_c(t - 1) + (1 - w(t)) \times S(t) \quad (17)$$

Now go back to Step1!

2 The Model

Concerning the model presented in this article, we suppose continuous re-balancing between the risky CDS portfolio and the cash account. Hence at any time the risky exposure is driven by the optimal leverage. As

$$L_{opt} = \frac{TBP(t) - NAV(t)}{TBP(t)} \times G = \frac{SF(t)}{TBP(t)} \times G \quad (18)$$

One can write

$$RE : L(t) \times TBP(t) = SF(t) \times G = G(TBP(t) - NAV(t)) \quad (19)$$

Similar to a CPPI strategy, we define a Cushion C and interpret the risk-less cash account as a risk-less zero-coupon bond. Hence the NAV takes the function of a floor:

$$C : V_t - B_t \rightarrow C_t = TBP(t) - NAV(t) \quad (20)$$

$$TBP(t) = SF(t) + NAV(t) \quad (21)$$

Obviously, the cushion corresponds to the shortfall. Since a CPDO structure is subject to the opposite CPPI rules, the strategy can be resumed as follows:

- If $TBP(t) > NAV(t) \Rightarrow RE: L \times TBP = G \times SF = G(TBP - NAV), G > 1$
- If $TBP(t) < NAV(t) \Rightarrow$ Cash-In event \rightarrow invest everything in r

We now apply the reasoning of R. Cont and P. Tankov [2006] within the CPPI framework to a CPDO, we can write the dynamics of the shortfall as follows:

$$\frac{dSF_t}{SF_t} = (G\mu^{TBP} - G\mu^{NAV} + r)dt + \sigma_t G dW_t \quad (22)$$

$$= (G \times \mu^{TBP} - Gr - Gr + r) dt + G\sigma_t dW_t \quad (23)$$

$$= (G \times \mu^{TBP} - 2Gr + r) dt + G\sigma_t dW_t \quad (24)$$

$$= [G \times TBP + (1 - 2G)r] dt + G\sigma_t dW_t \quad (25)$$

$$\Rightarrow SF_T = SF_0 \times e^{\left(rT + G(\mu^{TBP} - 2r)T + G\sigma_T W_T - \frac{G^2 \sigma_T^2 T}{2}\right)} \quad (26)$$

with

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_t \quad (27)$$

$$NAV_t = S_t + Account_t \quad (28)$$

under the risk-neutral probability measure Q .

The strategy of an investor who buys a CPDO tranche can be analyzed in the following manner:

Instead of investing in a tranche (and thus buying the TBP), he could invest an amount \tilde{N} in the NAV (like buying a kind of fund; combination of CDS portfolio and bonds) and make with the remaining money a bet on the shortfall. Consequently he buys $V_t = SF_t + NAV_t$. If the investor is lucky and the shortfall became reduced until maturity, the notional of SF_t will be incorporated in the Cash Account at time horizon T .

Proposition 2.1. *If $\mu^{TBP} > 2r$, the expected return of the CPDO global portfolio can be increased without any limit as one can write*

$$E_Q[V_T] = N + [TBP_0 - Ne^{-rT}] e^{rT+G(\mu^{TBP}-2r)T} \quad (29)$$

Proof. Suppose we invest \tilde{N} in the CDO Portfolio (i.e. the NAV).

Consequently \tilde{N} substitutes $S + Acc$ and one can write

$$TBP_T = SF_T + NAV_T = (TBP_T - NAV_T) + NAV_T \quad (30)$$

$$= \tilde{N} + (TBP_0 - \tilde{N}e^{-rt}) \times e^{\left((rT+G(\mu^{TBP}-2r)T+\sigma_T GW_T - \frac{G^2\sigma_T^2 T}{2})\right)} \quad (31)$$

$$E_Q[V_T] = E_Q[SF_T + NAV_T] \quad (32)$$

The rest follows. □

However in real world, there is no continuous re-balancing and even defaults occur. Hence the value of V_t is no longer guaranteed. In order to reflect this fact in our modeling approach, it is wiser to model the CDS spread rate with jumps:

2.1 Adding Jumps: The Setup

For comprehension's sake, we will model jumps with a standard Lévy process in order to attract the attention on the CPDO mechanism rather than mathematical issues. The more realistic Merton diffusion framework is presented in the Appendix.

The table below summarizes all assumptions:

$\{N_t\}_{t>0}$	Poisson Process
Z_i	Random Variable iid
Y_i	Random Variable iid
γ	Jump Intensity (average of γ jumps per unit of time)
$N^{A,B}$	Initial Notional of Tranche $[A; B]$
$X^{A,B}$	Outstanding Notional of Tranche $[A; B]$
μ^{TBP}	Expected Target Bond Price

Further we define

$$S \sim > \text{Lévy process} \quad (33)$$

$$S_t = S_0 \times e^{(r-\beta\gamma)t} \times \prod_{i=1}^{N_t} (Z_i + 1) \quad (34)$$

$$E[Z_i] = \beta \quad (35)$$

$$Z_i \perp N_t \quad (36)$$

$$Z_i + 1 = e^{Y_i} \quad (37)$$

$$E[e^{Y_i}] = \beta + 1 \quad (38)$$

$$K_t = \sum_{i=1}^{N_t} Y_i \quad (39)$$

$$\ln\left(\frac{S_t}{S_0}\right) = (r - \beta\gamma)t + \sum_{i=1}^{N_t} Y_i \quad (40)$$

$$r_t = r_0 \times e^{\int_0^t r_s ds} \quad (41)$$

$$S_t = S_0 \times e^{(r-\beta\gamma)t + \sum_{i=1}^{N_t} Y_i} \quad (42)$$

Also let $Z_t > -1$ which guarantees the positivity of the asset price.

Now the dynamics of the optimal Risky Exposure (i.e. the post-balanced short-fall) can be written as

$$SF_t = SF_0 \times e^{(G\mu^{TBP} - G(r - \beta\gamma + r) + r)t + G \sum_{i=1}^{N_t} Y_i} \quad (43)$$

Hence

$$RE_t^{opt} = RE_0^{opt} \times e^{(G\mu^{TBP} - G(r - \beta\gamma + r) + r)t + G \sum_{i=1}^{N_t} Y_i} \quad (44)$$

Now we already mentioned previously that the Net Asset value (NAV) is subject to Cash-In / Cash-out conditions. The Cash-In time is defined as

$$\tau^{in} = \inf\{t : NAV_t > TBP\} = \inf\{t : RE_t^{opt} + B(0, T) > TBP\} \quad (45)$$

In consequence the Cash-Out time is presented as

$$\tau^{out} = \inf\{t : NAV_t < Threshold\} = \inf\{t : RE_t^{opt} + B(0, T) < Threshold\} \quad (46)$$

2.2 The outstanding Tranche's Notional

The outstanding tranche's notional is a central element when it comes to pricing securitized tranches. The structure's deal spread (i.e. the [%]-spread pre-determined in the term sheet) is paid on the tranche's notional. However the latter can diminish via credit events in the case of CDO structures. In our CPDO case, we consider that the tranche's notional is potentially reduced by the prevailing shortfall:

$$E_Q [X^{A,B}(t) \mid \tau^{in} > t, \tau^{out} > t] \quad (47)$$

$$= E_Q \left[[N^{A,B} - [SF_t - A]^+]^+ \mid \tau^{in} > t, \tau^{out} > t \right] \quad (48)$$

Obviously the difficulty consists in calculating $E_Q [(SF_t - A)^+ \mid \tau^{in} > t, \tau^{out} > t]$ which can be interpreted as a *double barrier option*, more precisely a double barrier Call written on the shortfall with the tranche's Attachment Point as strike.

Let TS denote the CPDO structure's Cash-Out Threshold and one can state

$$E_Q [X^{A,B}(t) \mid \tau^{in} > t, \tau^{out} > t] \quad (49)$$

$$= E_Q [X^{A,B}(t) \mid \min(RE_t^{opt}) > TBP - B(0, t), \max(RE_t^{opt}) < TS - B(0, t)] \quad (50)$$

We substitute the upper and the lower barrier with the following variables:

$$D = TBP - B(0, t)$$

$$U = TS - B(0, t)$$

This gives

$$E_Q [X^{A,B}(t) \mid \tau^{in} > t, \tau^{out} > t] \quad (51)$$

$$= E_Q [X^{A,B}(t) \mid \min(RE_t^{opt}) > D, \max(RE_t^{opt}) < U] \quad (52)$$

$$= \frac{E_Q [X^{A,B}(t) 1_{\{\min(RE_t^{opt}) > D \cap \max(RE_t^{opt}) < U\}}]}{Q(\min(RE_t^{opt}) > D \cap \max(RE_t^{opt}) < U)} \quad (53)$$

In the following we present two manners for calculating the above expression:

2.2.1 The Expected Conditional Shortfall

The central term we have to calculate is

$$E_Q [SF_t 1_{\{\min(RE_t^{opt}) > D \cap \max(RE_t^{opt}) < U\}}]$$

This can be done as follows:

$$E_Q [SF_t 1_{\{\min(RE_t^{opt}) > D \cap \max(RE_t^{opt}) < U\}}] \quad (54)$$

$$= E_Q \left[SF_0 e^{(G\mu^{TBP} - G(r - \beta\gamma + r) + r)t + G \sum_{i=1}^{N_t} Y_i} 1_{\underbrace{\{\min(RE_t^{opt}) > D \cap \max(RE_t^{opt}) < U\}}_{\Xi}} \right] \quad (55)$$

$$= \sum_{n=0}^{\infty} E_Q [SF_0 e^{(G\mu^{TBP} - G(r - \beta\gamma + r) + r)t + G \sum_{i=1}^n Y_i} 1_{\Xi} \mid N_t = n] P(N_t = n) \quad (56)$$

Recall that

$$e^x = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$$

$$P(N_T = k) = \frac{(\gamma T)^k}{k!} \times e^{-\gamma T}$$

$$Y_i \hookrightarrow N(\alpha, \delta^2)$$

$$\sum_{i=1}^n Y_i \hookrightarrow n\alpha + \sqrt{n\delta^2}z$$

$$z \hookrightarrow N(0, 1)$$

Hence

$$\sum_{n=0}^{\infty} E_Q \left[SF_0 e^{(G\mu^{TBP} - G(r - \beta\gamma + r) + r)t + G \sum_{i=1}^n Y_i} 1_{\{\Xi\}} \mid N_t = n \right] P(N_t = n) \quad (57)$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma t)^n}{n!} e^{-\gamma t} E_Q \left[SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + G(n\alpha + \sqrt{n\delta^2}z)} 1_{\{\Xi\}} \right] \quad (58)$$

The double barrier no-touch option problem is well-known and exhaustively treated in Carr [2002]. The problem of evaluating the pay-off can be solved via the reflection theorem. This approach consists in reflecting the option's pay-off region at each barrier and to sum up over the marginal reflected pay-offs. The reader who is interested in detail how the reflection theorem works out should refer to the appendix.

In analogy to Carr & Chou [2002] we consider a double barrier no-touch option with the following characteristics:

The option is written on an asset SF (the shortfall), we refer to the upper barrier with U and to the lower barrier with D . The pay-off function of the shortfall within a double no-touch binary for impair regions can be written as follows:

We define

$$\tilde{\mu} := (G\mu^{TBP} - G(2r - \beta\gamma) + r)t + Gn\alpha$$

Further

$$\left[SF_t > \frac{U^k}{D^{k-1}} \right] - \left[SF_t > \frac{U^{k+1}}{D^k} \right] \quad (59)$$

In particular

$$\left[SF_t > \frac{U^k}{D^{k-1}} \right] = \left[SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + G(n\alpha + \sqrt{n\delta^2}z)} > \frac{U^k}{D^{k-1}} \right] \quad (60)$$

$$\Rightarrow z < \underbrace{\frac{\ln\left(\frac{SF_0 D^{k-1}}{U^k}\right) - (G\mu^{TBP} - G(2r - \beta\gamma) + r)t - Gn\alpha}{G\sqrt{n\delta^2}}}_{=: d_1} \quad (61)$$

$$d_1 = \frac{\ln\left(\frac{SF_0 D^{k-1}}{U^k}\right) - \tilde{\mu}}{G\sqrt{n\delta^2}} \quad (62)$$

Consequently

$$\left[SF_t > \frac{U^{k+1}}{D^k} \right] = \left[SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + G(n\alpha + \sqrt{n\delta^2}z)} > \frac{U^{k+1}}{D^k} \right] \quad (63)$$

$$\Rightarrow z < \underbrace{\frac{\ln\left(\frac{SF_0 D^k}{U^{k+1}}\right) - (G\mu^{TBP} - G(2r - \beta\gamma) + r)t - Gn\alpha}{G\sqrt{n\delta^2}}}_{=:d_2} \quad (64)$$

$$d_2 = \frac{\ln\left(\frac{SF_0 D^k}{U^{k+1}}\right) - \tilde{\mu}}{G\sqrt{n\delta^2}} \quad (65)$$

As far as pair regions are concerned the pay-off function is

$$\left[SF_t < \frac{U^{k+1}}{D^k} \right] - \left[SF_t < \frac{U^k}{D^{k-1}} \right] \quad (66)$$

In particular

$$\left[SF_t < \frac{U^{k+1}}{D^k} \right] \Rightarrow z < N \left(\frac{-\ln\left(\frac{SF_0 D^k}{U^{k+1}}\right) - \tilde{\mu}}{G\sqrt{n\delta^2}} \right) \quad (67)$$

$$\left[SF_t < \frac{U^k}{D^{k-1}} \right] \Rightarrow z < N \left(\frac{-\ln\left(\frac{SF_0 D^{k-1}}{U^k}\right) - \tilde{\mu}}{G\sqrt{n\delta^2}} \right) \quad (68)$$

$$(69)$$

Thus

$$\left[SF_t < \frac{U^{k+1}}{D^k} \right] - \left[SF_t < \frac{U^k}{D^{k-1}} \right] \quad (70)$$

$$= N \left(\frac{-\ln\left(\frac{SF_0 D^k}{U^{k+1}}\right) - \tilde{\mu}}{G\sqrt{n\delta^2}} \right) - N \left(\frac{-\ln\left(\frac{SF_0 D^{k-1}}{U^k}\right) - \tilde{\mu}}{G\sqrt{n\delta^2}} \right) \quad (71)$$

$$= N \left(\frac{\ln\left(\frac{SF_0 D^{k-1}}{U^k}\right) + \tilde{\mu}}{G\sqrt{n\delta^2}} \right) - N \left(\frac{\ln\left(\frac{SF_0 D^k}{U^{k+1}}\right) + \tilde{\mu}}{G\sqrt{n\delta^2}} \right) \quad (72)$$

$$= N(\tilde{d}_1) - N(\tilde{d}_2) \quad (73)$$

with

$$d_{1/2}^{\sim} = d_{1/2} + \frac{2\mu}{\sqrt{n\delta^2}}$$

By applying Carr's formula, the marginal pay-offs for pair and impair regions can be written as

$$V(SF, k) = \left\{ \begin{array}{ll} -\left(\frac{SF}{U}\right)^p \times \left(\frac{D}{U}\right)^{jp} [N(d_1) - N(d_2)] & \text{in region } k = 2j + 1 \\ \left(\frac{U}{P}\right)^{jp} [N(\tilde{d}_1) - N(\tilde{d}_2)] & \text{in region } k = 2j \end{array} \right\} \quad (74)$$

where j is an integer.

Finally

$$E_Q [SF_t 1_{\{\min(SF_t) > D \cap \max(SF_t) < U\}}] = \sum_{n=0}^{\infty} \frac{(\gamma t)^n}{n!} e^{-\gamma t} \sum_{k=-\infty}^{\infty} V(SF, k) \quad (75)$$

Now $Q(\min(SF_t) > D \cap \max(SF_t) < U)$ is obtained the same way and with (51)

we have $E_Q [SF_t | \tau^{in}, \tau^{out} > t]$.

The outstanding tranche's notional of tranche $[A; B]$, which is

$$E_Q [X^{A,B}(t) | \tau^{in} > t, \tau^{out} > t] \quad (76)$$

$$= E_Q \left[[N^{A,B} - [SF_t - A]^+ | \tau^{in} > t, \tau^{out} > t] \right] \quad (77)$$

$$= E_Q \left[[(B - A) - [SF_t - A]^+ | \tau^{in} > t, \tau^{out} > t] \right] \quad (78)$$

$$= E_Q \left[[(B - SF_t) 1_{\{A < SF_t < B\}}]^+ | \tau^{in} > t, \tau^{out} > t] \right] \quad (79)$$

$$= B - E_Q \left[[SF_t 1_{\{A < SF_t < B\}}]^+ | \tau^{in} > t, \tau^{out} > t] \right] \quad (80)$$

which can be calculated by simple integration.

2.2.2 The Barrier Call Spread Approach

When we model the CPDO structure's shortfall, this is equivalent to modeling the structure's virtual implied loss. Now a CPDO tranche (or also a CDO tranche) can be interpreted as a Call Spread on the structure's implied loss with the respective Attachment/Detachment Point as strikes. Hence one can write

$$E_Q [X^{A,B}(t) 1_{\{\min(SF_t) > D \cap \max(SF_t) < U\}}] = \quad (81)$$

$$= E_Q [(SF_t - A)^+ 1_{\{\min(SF_t) > D\}}] - E_Q [(SF_t - B)^+ 1_{\{\max(SF_t) > U\}}] \quad (82)$$

We focus on the expression $E_Q [(SF_t - A)^+ 1_{\{\min(SF_t) > D\}}]$ first:

$$E_Q [(SF_t - A)^+ 1_{\{\min(SF_t) > D\}}] \quad (83)$$

$$= E_Q [SF_t 1_{\{\min(SF_t) > D\}} 1_{\{SF_t > A\}}] - AE_Q [1_{\{\min(SF_t) > D\}} 1_{\{SF_t > A\}}] \quad (84)$$

Fortunately within the traditional Black & Scholes diffusion framework there exists the following formula for this type of barrier call option:

$$P(x_t > x, y_t > y) = N\left(\frac{-x + \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-x + 2y + \mu T}{\sigma\sqrt{T}}\right)$$

with

$$y_t = \inf(x_t), x \geq y$$

where $\mu = (r - \frac{\sigma^2}{2})T$ while the asset S follows the dynamics $\frac{dS_t}{S_t} = r + \sigma_t dW_t$ under the risk-neutral probability measure.

Obviously this formula is obtained via the reflection theorem. This has already been illustrated in numerous articles.

The adaption to our Lévy Process framework gives:

$$N\left(\frac{-x + \mu T}{\sigma\sqrt{T}}\right) \rightarrow N\left(\frac{-A - (G(2r - \beta\gamma) + r)t + Gn\alpha}{\underbrace{G\sqrt{n\delta^2}}_{:=d_1^*}}\right) \quad (85)$$

$$N\left(\frac{-x + 2y + \mu T}{\sigma\sqrt{T}}\right) \rightarrow N\left(\frac{-A + 2D - (G(2r - \beta\gamma) + r)t + Gn\alpha}{\underbrace{G\sqrt{n\delta^2}}_{:=d_2^*}}\right) \quad (86)$$

Thus

$$P(SF_t > A, \inf(SF_t) > D) = N(d_1^*) - e^{\frac{2\mu D}{G^2 n \delta^2}} N(d_2^*)$$

Besides

$$\begin{aligned} E_Q [SF_t 1_{\{\min(SF_t) > D\}} 1_{\{SF_t > A\}}] &= \quad (87) \\ &= \underbrace{\sum_{n=0}^{\infty} \frac{(\gamma t)^n}{n!} e^{-\gamma t}}_{:=\Gamma} E_Q [SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + G(n\alpha + \sqrt{n\delta^2}z)} 1_{\{\min(SF_t) > D\}} 1_{\{SF_t > A\}}] \end{aligned}$$

$$= \Gamma \left[\int_{-d_1^*}^{\infty} SF_0 e^{\tilde{\mu} + G\sqrt{n\delta^2}z} e^{-\frac{z^2}{2}} \frac{dz}{2\pi} - e^{\frac{2\tilde{\mu}D}{G^2 n \delta^2}} \int_{-d_2^*}^{\infty} SF_0 e^{\tilde{\mu} + G\sqrt{n\delta^2}z} e^{-\frac{z^2}{2}} \frac{dz}{2\pi} \right] \quad (89)$$

$$= \Gamma \left[SF_0 e^{\tilde{\mu}} \int_{-d_1^*}^{\infty} e^{G\sqrt{n\delta^2}z} e^{-\frac{z^2}{2}} \frac{dz}{2\pi} - e^{\frac{2\tilde{\mu}D}{G^2 n \delta^2}} SF_0 e^{\tilde{\mu}} \int_{-d_2^*}^{\infty} e^{G\sqrt{n\delta^2}z} e^{-\frac{z^2}{2}} \frac{dz}{2\pi} \right] \quad (90)$$

$$= \Gamma \left[SF_0 e^{\tilde{\mu} - G^2 \frac{n\delta^2}{2}} \int_{-d_1^*}^{\infty} e^{-\frac{(z - G\sqrt{n\delta^2})^2}{2}} \frac{dz}{2\pi} - e^{\frac{2\tilde{\mu}D}{G^2 n \delta^2}} SF_0 e^{\tilde{\mu} - G^2 \frac{n\delta^2}{2}} \int_{-d_2^*}^{\infty} e^{-\frac{(z - G\sqrt{n\delta^2})^2}{2}} \frac{dz}{2\pi} \right] \quad (91)$$

With

$$y =: z - \sqrt{n\delta^2} \Rightarrow z = y + \sqrt{n\delta^2}$$

we can write

$$\int_{-d_1^*}^{\infty} e^{-\frac{(z-G\sqrt{n\delta^2})^2}{2}} \frac{dz}{2\pi} = \int_{-\infty}^{d_1^{**}} e^{-\frac{y^2}{2}} \frac{dy}{2\pi}$$

where

$$d_1^{**} = d_1^* + G\sqrt{n\delta^2}$$

The same applies to d_2^* and d_1^{**} .

Hence

$$E_Q [SF_t \mathbf{1}_{\{\min(SF_t) > D\}} \mathbf{1}_{\{SF_t > A\}}] = \quad (92)$$

$$= \Gamma \left[SF_0 e^{\tilde{\mu} - G^2 \frac{n\delta^2}{2}} \int_{-\infty}^{d_1^{**}} e^{-\frac{y^2}{2}} \frac{dy}{2\pi} - e^{\frac{2\tilde{\mu}D}{G^2 n\delta^2}} SF_0 e^{\tilde{\mu} - G^2 \frac{n\delta^2}{2}} \int_{-\infty}^{d_2^{**}} e^{-\frac{y^2}{2}} \frac{dy}{2\pi} \right] \quad (93)$$

It follows that

$$E_Q [(SF_t - A)^+ \mathbf{1}_{\{\min(SF_t) > D\}}] \quad (94)$$

$$= E_Q [SF_t \mathbf{1}_{\{\min(SF_t) > D\}} \mathbf{1}_{\{SF_t > A\}}] - A E_Q [\mathbf{1}_{\{\min(SF_t) > D\}} \mathbf{1}_{\{SF_t > A\}}] \quad (95)$$

$$= \Gamma \left[SF_0 e^{\tilde{\mu} - G^2 \frac{n\delta^2}{2}} \int_{-\infty}^{d_1^{**}} e^{-\frac{y^2}{2}} \frac{dy}{2\pi} - e^{\frac{2\tilde{\mu}D}{G^2 n\delta^2}} SF_0 e^{\tilde{\mu} - G^2 \frac{n\delta^2}{2}} \int_{-\infty}^{d_2^{**}} e^{-\frac{y^2}{2}} \frac{dy}{2\pi} \right] \quad (96)$$

$$- A \times \left[\int_{-d_1^*}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{2\pi} - e^{\frac{2\tilde{\mu}D}{G^2 n\delta^2}} \int_{-d_2^*}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{2\pi} \right] \quad (97)$$

Now let us have a closer look on $E_Q [(SF_t - B)^+ \mathbf{1}_{\{\max(SF_t) > U\}}]$:

$$E_Q [(SF_t - B)^+ \mathbf{1}_{\{\max(SF_t) > U\}}] = \quad (98)$$

$$= E_Q [SF_t \mathbf{1}_{\{\max(SF_t) > U\}} \mathbf{1}_{\{SF_t > B\}}] - B E_Q [\mathbf{1}_{\{\max(SF_t) > U\}} \mathbf{1}_{\{SF_t > B\}}] \quad (99)$$

$$(100)$$

There does also exist a formula within the Black & Scholes diffusion framework:

$$P(x_t \geq x, Y_t \geq y) = P(x_t \geq x) - P(x_t \geq x, Y_t \leq y) \quad (101)$$

$$= P(x_t \geq x) - \left[N\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{x - 2\mu - \mu t}{\sigma\sqrt{t}}\right) \right] \quad (102)$$

The transformation of this formula into our Lévy Process framework is done in analogy of what we have demonstrated previously and the calculation of the expectations is straightforward.

Finally we obtain a formula for the outstanding tranche's notional

$E_Q [X^{A,B}(t) \mathbf{1}_{\{\min(SF_t) > D \cap \max(SF_t) < U\}}]$. The conditional form

$E_Q [X^{A,B}(t) \mid \tau^{in}, \tau^{out} > t]$ is obtained by applying formula (51).

2.3 The Expected Cash Flow on outer tranches

Comparable to Dorn [2007] we consider in the following a *CPDO* tranche which is characterized by its Attachment Point $A\%$ and its Detachment Point $B\%$, both referring to the total notional of issued notes.

As for CDOs, pricing a *CPDO* tranche is very similar to pricing a basket of credit default swaps. In case of *CPDO*, the shortfall of the entire CPDO structure can be interpreted as a virtual cumulative loss which may erode the tranche's subordination and affect the tranche.

First, we have to estimate the present value of tranche losses triggered by the shortfall during the *CPDO* lifetime.

Second, we calculate the present value of the premium payments weighted by the outstanding capital (original tranche amount minus shortfall).

The former is called the *Protection Leg* and we refer to the latter as *Premium Leg*.

The fair spread s is defined such that the expected value of both legs is equal (no free lunch). The expectation is taken under the risk-neutral measure:

$$s = \frac{E_Q[\text{Protectionleg}]}{E_Q[\text{Premiumleg}]}$$

2.3.1 The Premium Leg:

The protection buyer pays a regular fixed spread (premium) on the outstanding notional of the tranche (premium leg).

We refer to the outer tranche's outstanding notional with $X^{A,B}(t)$.

Assume that the payment dates are fixed in regular intervals until the end of the *CPDO* life cycle. Hence these dates can be considered as

$$(t_1, \dots, t_i, \dots, t_p), \text{ for } i = 1, \dots, p.$$

The expression for the Premium Leg therefore is:

$$\text{Premiumleg} = s \times \sum_{i=1}^p E_Q [X^{A,B}(t_i)] \times e^{-rt_i} \times \delta(t_{i-1}, t_i) + \text{Accrual}$$

In most cases however, the default time of a credit name is situated between two regular spread payment dates. In this case, the spread payment has to be accrued

with respect to the elapsed time between the last regular payment date and the default time.

Accrual:

$$s \times \int \delta(t_{(i-1)}, t_i) e^{-r(t+\Delta t)} dE_Q[X^{A,B}(t)] \approx \sum_{i=1}^n \dots (E_Q[X^{A,B}(t_{i-1})] - E_Q[X^{A,B}(t_i)])$$

with Δt representing the accrued time.

2.3.2 The Protection Leg

The protection seller pays a single payment on each default that affects and reduces the tranche. Hence the Protection Leg or Default Leg is calculated as follows:

$$\int e^{-r(t+\Delta t)} dE_Q[X^{A,B}(t)] \approx \sum_{i=1}^p e^{-rt_i} \times (E_Q[X^{A,B}(t_{i-1})] - E_Q[X^{A,B}(t_i)])$$

The above can be written as any variation on the outstanding outer tranche notional between periods $i - 1$ and i would indicate a loss.

By discounting the spread paid on each *CPDO* tranche with respect to the *CPDO* life cycle, one can now evaluate any tranche of a *CPDO*!

3 Simulation Results

We simulated a *CPDO* structure based on iTraxx IG 5yr quotes [Exhibit 4-11]. Notably Exhibit 5 and Exhibit 6 illustrate that optimal re-balancing allows for an earlier Cash-In Event (plain surface).

4 Conclusion

In this article we demonstrated how the opposite CPPI mechanism interacts with the traditional CPDO structure. However this engenders that all risks inherent to CPPIs also apply to a CPDO structure. Gap Risk -just to name the most important one- is present as explained briefly in the introductory section of this work. We explained that when the NAV hits the lower threshold, the deal will be unwound. But in a sudden market downside movement, the guaranteed capital could be inferior as initially forecasted. Those problems are not quantified in this work yet but the author plans to do so soon. This illustrates that the investor should be aware of the dangers which bear new asset classes. CPDOs are the first reaction of investment banks to limit the risks of traditional CDOs by employing capital guaranteeing features. The next step will consist in drawing the popular securitization techniques away from the risky credit market by applying them to other asset classes. This is the case of CFXOs which make use of CDO technology based on the FX markets. The author will also present a paper on this subject soon.

5 Endnotes

The author would like to thank Professor Patrice Poncet for his contribution and expresses his gratitude to Sergey Kas'yan for fruitful discussions about jump processes.

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6 Appendix

The Reflection Theorem

Theorem 6.1. The Reflection Theorem: *In a Black-Scholes economy, suppose that X is a portfolio of European options expiring at time T with payoff:*

$$X(S_T) = \left\{ \begin{array}{ll} f(S_T) & \text{if } S_T \in (U, D) \\ 0 & \text{otherwise.} \end{array} \right\}$$

For $H > 0$, let Y be a portfolio of European options with maturity T and pay-off:

$$Y(S_T) = \left\{ \begin{array}{ll} \left(\frac{S_T}{H}\right)^p f(H^2/S_T) & \text{if } S_T \in (H^2/B, H^2/A) \\ 0 & \text{otherwise.} \end{array} \right\}$$

where the power $p = 1 - \frac{2r}{\sigma^2}$ and r, σ are the interest rate and volatility respectively.

Then X and Y have the same value whenever the spot equals H .

Consider a double knockout option with down barrier D , up barrier U , and maturity date T . We begin by dividing the interval $[0, \infty]$ into regions as in Figure below. We can succinctly define the regions as:

$$\text{Region } k = \left[\left(\frac{U}{D}\right)^k D, \left(\frac{U}{D}\right)^k U \right]$$

The adjusted pay-off for a region i will be described by

$$F_i(S_t)$$

We start with $f_0(S_t) = f(S_t)$ where f represents the pay-off function of the asset S within the traditional Black & Scholes framework. The reflection theorem states that if one reflects a region k along the barrier D , this would result in region $-k-1$. Similarly, for reflection along U , region k would be the reflection of region $-k+1$. It is useful to define the following two operators:

$$R_D(f(S_T)) = - \left(\frac{S_T}{D}\right)^p f(D^2/S_T)$$

and

$$R_U(f(S_T)) = - \left(\frac{S_T}{U}\right)^p f(U^2/S_T)$$

Consequently

$$f_k(S_T) = R_D(f_{-k-1}(S_T)), \text{ for } k < 0$$

$$f_k(S_T) = R_U(f_{-k-1}(S_T)), \text{ for } k > 0$$

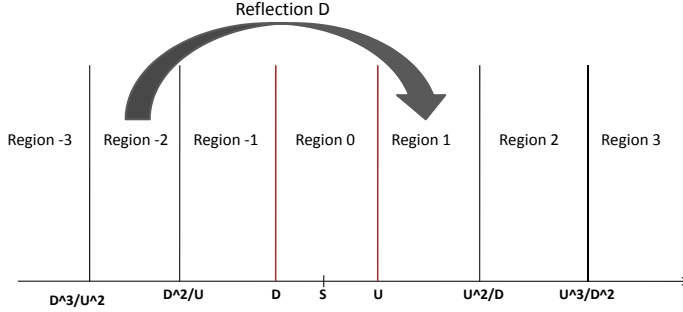


Figure 2:

R_U and R_D bijectively map between the corresponding regions. Also, we are taking the negative of the reflection, so that the valuation of the payoffs will cancel. By induction, we can completely determine the entire adjusted payoff as:

$$f_k(S_T) = \left\{ \begin{array}{ll} f(S_T) & \text{for } k = 0 \\ \underbrace{R_D \circ R_U \circ R_D \cdots}_{k \text{ operators}}(f(S_T)) & \text{for } k < 0 \\ \underbrace{R_U \circ R_D \circ R_U \cdots}_{k \text{ operators}}(f(S_T)) & \text{for } k > 0 \end{array} \right\} \quad (103)$$

A portfolio of European options that delivers the above adjusted payoff replicates the payoff to a double barrier claim. If we never touch either barrier, then the adjusted payoff from region 0 matches the payoff of the original exotic. Upon reaching a barrier, the values of the payoff above the barrier are canceled by the value of the payoff below the barrier. Therefore, our portfolio is worth zero at either barrier at which point we can liquidate our position. To find the adjusted payoff for a one-touch claim, we apply in-out parity. The adjusted payoff is given by:

$$f_k(S_T) = \left\{ \begin{array}{ll} 0 & \text{for } k = 0 \\ f(S_T) - \underbrace{R_D \circ R_U \circ R_D \cdots}_{k \text{ operators}}(f(S_T)) & \text{for } k < 0 \\ f(S_T) - \underbrace{R_U \circ R_D \circ R_U \cdots}_{k \text{ operators}}(f(S_T)) & \text{for } k > 0 \end{array} \right\} \quad (104)$$

To compute the price of the double no-touch binary, we simply compute the price of the adjusted payoff in each region and sum over all regions. The price can be found by taking the discounted expected value of the payoff under the risk-neutral measure. If the current spot price is S , the value of the payoff in region k is:

$$\Pi(SF, k) = \left\{ \begin{array}{ll} -\left(\frac{S}{U}\right)^p \times \left(\frac{D}{U}\right)^{jp} \left[N\left(\frac{\ln(x_1) - \nu T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln(x_2) - \nu T}{\sigma\sqrt{T}}\right) \right] & \text{in region } k = 2j + 1 \\ \left(\frac{U}{P}\right)^{jp} \left[N\left(\frac{\ln(x_1) + \nu T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln(x_1) + \nu T}{\sigma\sqrt{T}}\right) \right] & \text{in region } k = 2j \end{array} \right\} \quad (105)$$

where j is an integer and $x_1 = \frac{SD^{k-1}}{U^k}$, $x_2 = \frac{SD^k}{U^{k+1}}$ and $\nu = r - 1/2\sigma^2$.

The value of the no-touch binary is the sum of the value for each region:

$$NTB(S) = \sum_{k=-\infty}^{\infty} \Pi(S, k)$$

6.1 The Merton Framework

The following calculation is meant to illustrate the handling of the Merton Model and can be applied easily to the calculations presented in the article. We consider the expression

$$E_Q [SF_t 1_{\{SF_t \geq B\}}]$$

where B represents a barrier threshold.

Within the Merton framework, the shortfall follows the dynamic

$$SF_t = SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + G \sum_{i=1}^{N_t} Y_i + G\sigma W_t - 1/2G^2\sigma^2 t} \quad (106)$$

With the previously defined properties and notations one can write

$$\begin{aligned} E_Q [SF_t 1_{\{SF_t \geq B\}}] & \quad (107) \\ &= \sum_{n=0}^{\infty} \frac{(\gamma t)^n}{n!} e^{-\gamma t} E_Q \left[SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + G \sum_{i=1}^{N_t} Y_i + G\sigma W_t - 1/2G^2\sigma^2 t} 1_{\{SF_t \geq B\}} \right] \end{aligned} \quad (108)$$

Recall that

$$\begin{aligned} \sigma W_t &\hookrightarrow \sigma\sqrt{t}z \\ az_1 + bz_2 &\hookrightarrow \sqrt{a^2 + b^2}z \Rightarrow \sqrt{n\delta^2}z_1 + \sigma\sqrt{t}z_2 \hookrightarrow \sqrt{n\delta^2 + \sigma^2 t} \times z \\ z &= z_1 + z_2 \end{aligned}$$

Thus

$$E_Q [SF_t 1_{\{SF_t \geq B\}}] \quad (109)$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma t)^n}{n!} e^{-\gamma t} E_Q \left[SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + G(\sqrt{n\delta^2 + \sigma^2 t} \times z + n\alpha) - 1/2G^2\sigma^2 t} 1_{\{SF_t \geq B\}} \right] \quad (110)$$

It follows that the event $SF_t > B$ can be re-written as

$$z < \underbrace{\frac{\ln\left(\frac{SF_0}{B}\right) - (G\mu^{TBP} - G(2r - \beta\gamma) + r)t - G(\sqrt{n\delta^2 + \sigma^2 t} \times z + n\alpha) + 1/2G^2\sigma^2 t}{G\sqrt{n\delta^2 + \sigma^2 t}}}_{:=d_0} \quad (111)$$

$$E_Q [SF_t 1_{\{SF_t \geq B\}}] \quad (112)$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma t)^n}{n!} e^{-\gamma t} SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + Gn\alpha - 1/2G^2\sigma^2 t} \int_{-d_0}^{\infty} e^{G\sqrt{n\delta^2 + \sigma^2 t} \times z} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \quad (113)$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma t)^n}{n!} e^{-\gamma t} SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + Gn\alpha + G^2 \frac{n\delta^2 + \sigma^2 t}{2}} \int_{-d_0}^{\infty} e^{-\frac{(z - G\sqrt{n\delta^2 + \sigma^2 t})^2}{2}} \frac{dz}{\sqrt{2\pi}} \quad (114)$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma t)^n}{n!} e^{-\gamma t} SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + Gn\alpha + G^2 \frac{n\delta^2 + \sigma^2 t}{2}} \int_{-\infty}^{\tilde{d}_1} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \quad (115)$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma t)^n}{n!} e^{-\gamma t} SF_0 e^{(G\mu^{TBP} - G(2r - \beta\gamma) + r)t + Gn\alpha + G^2 \frac{n\delta^2 + \sigma^2 t}{2}} N(\tilde{d}_1) \quad (116)$$

with $y = z - G\sqrt{n\delta^2 + \sigma^2 t}$ and $\tilde{d}_1 = d_0 + G\sqrt{n\delta^2 + \sigma^2 t}$.

7 Exhibit

Exhibit 1: Illustration of Gap Risk.

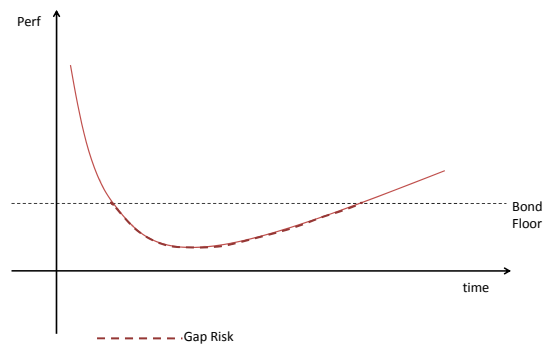


Exhibit 2: Net Asset Value Variations.

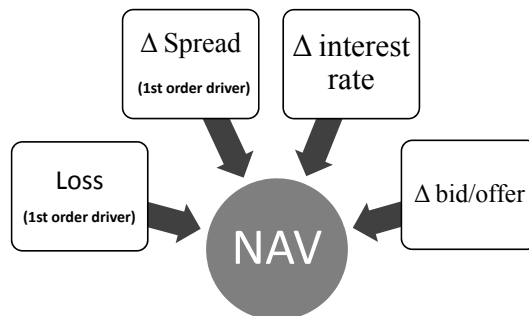


Exhibit 3: Target Notional Determinants.

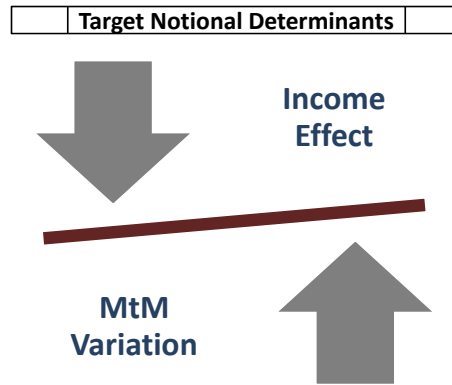


Exhibit 4: Quoted Market CDS Spreads vs. Contracted Spread

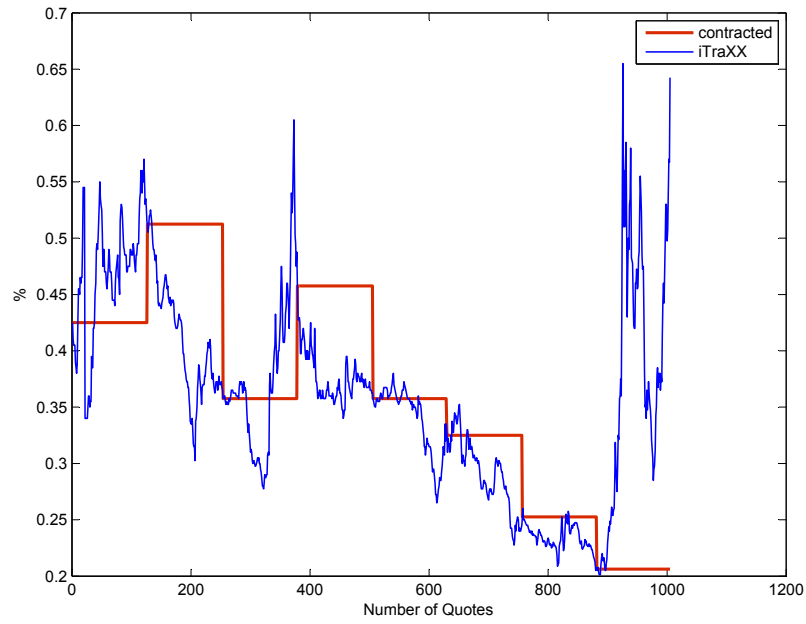


Exhibit 5: NAV - Leverage surface without const Re-balancing.

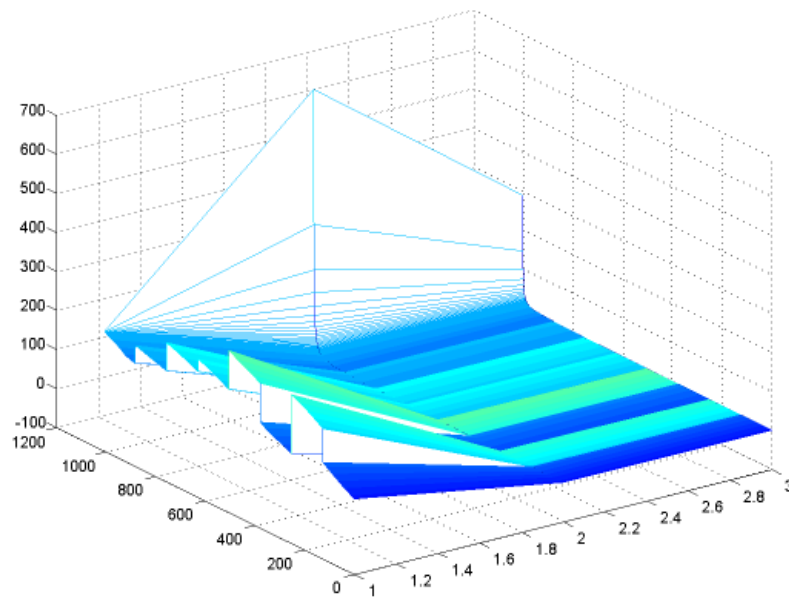


Exhibit 6: NAV - Leverage surface post Re-balancing.

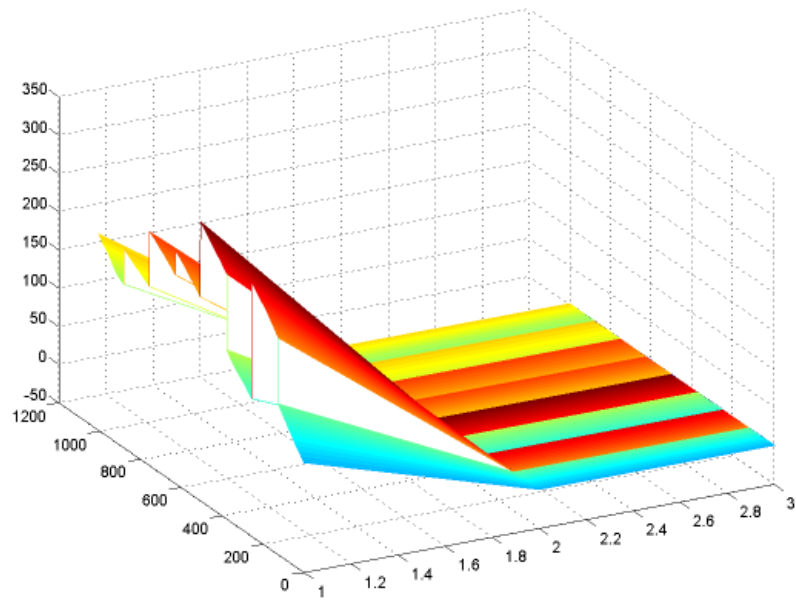


Exhibit 7: NAV - Continuous CPDO Performance drivers without const Re-balancing.

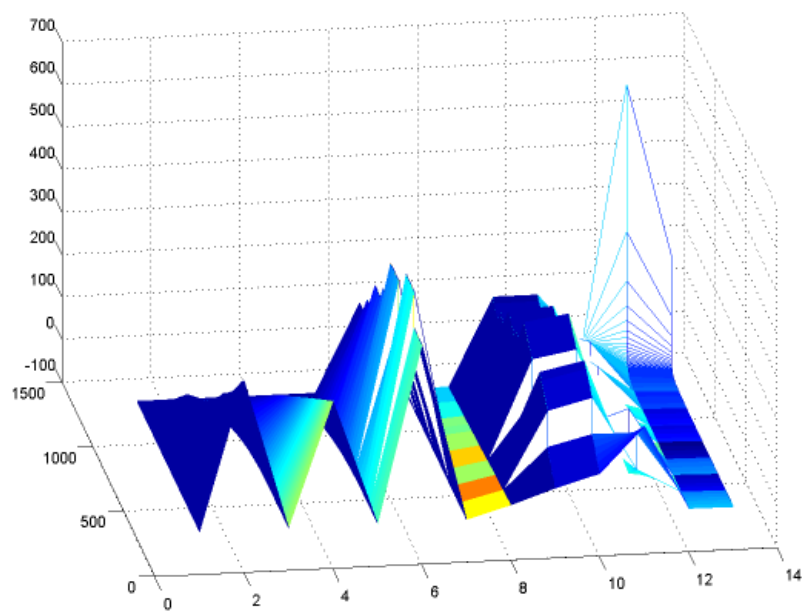


Exhibit 8: Re-balanced CPDO Performance Drivers.

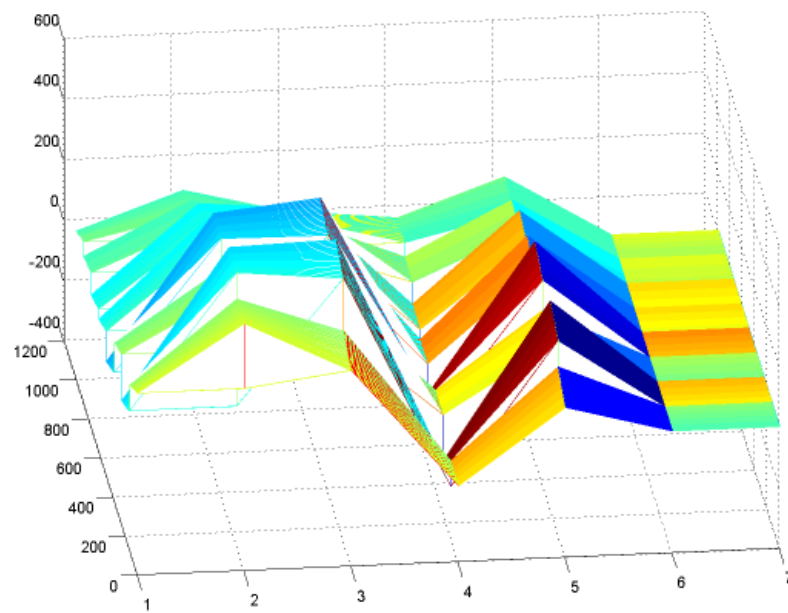


Exhibit 9: Re-balanced MtM, Shortfall, Cash Account converge via Cash-in Event.

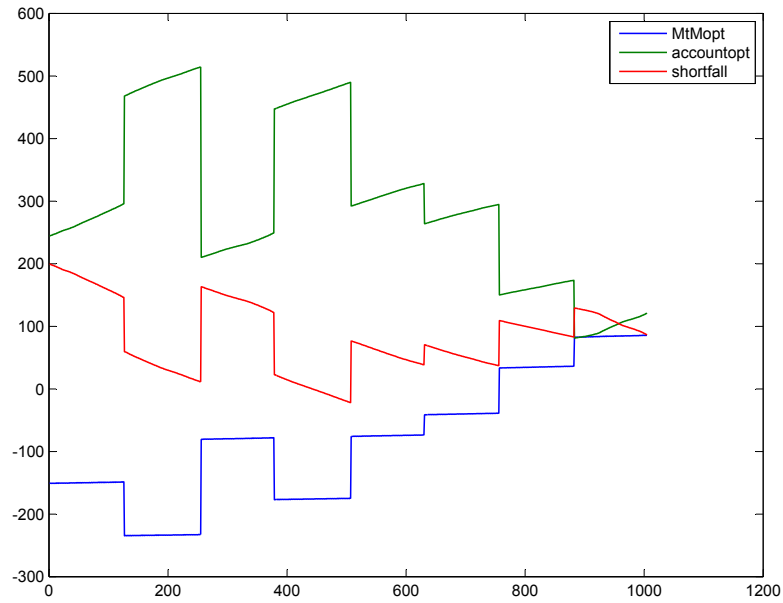


Exhibit 10: Opt. Leverage vs Impl. Leverage without const
Re-balancing.

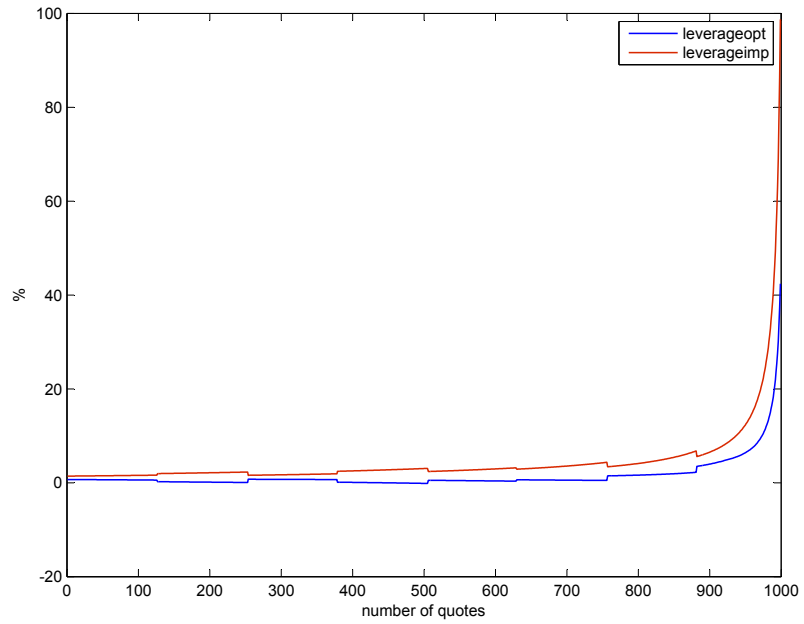


Exhibit 11: Opt. Leverage vs Impl. Leverage post Re balancing.

