Modelling of Surface Air Temperature and Pricing of Weather Derivatives

By

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Abstract-This study attempts to formulate a pricing model for the weather derivatives, whose payoffs depend on surface air temperature. Daily temperature data for the last thirty years is closely analyzed for four cities in U.K. to model a temperature process which captures the daily temperature fluctuations including the seasonal patterns and the year-on-year up-ward trend behaviour of the temperature. This work further evaluates an arbitrage-free option pricing using a Gaussian Ornstein-Uhlenbeck model. Keeping in mind that temperature, the underlying variable of the weather derivative, is non-tradable we consider a risk premium estimator to find the price of a weather derivatives contract. Finally, the study provides results based on these models as well as based on Monte Carlo Simulations.

Keywords: Weather derivatives, surface-air-temperature, Gaussian Ornstein-Uhlenbeck model, option pricing and Monte Carlo Simulations.

1. Introduction

The performances of many companies are exposed to the changes in weather. The industry sectors principally exposed to 'weather risk' includes basic materials, consumer durables and agricultural industries. Amongst these various sectors the basic materials has mainly triggered the demand for the weather derivatives market and the rapid growth in the weather risk assessment industry. Firms take several measures in different ways to prevent their exposure against adverse weather conditions. Some of these measures include: real and infrastructural investment (for example creation of water reservoirs to prevent industrial work from water scarcity during periods of low precipitation), investment in public insurance contracts and investment in private insurance and legal provisions (for example compulsory safety measures required for usage of land which are prone to natural calamities)

The weather derivatives is a fairly recent set of financial instrument introduced which enables the companies to hedge their risks related unfavourable uncertain weather conditions. The underlying asset of the weather derivative is a weather parameter which affects the volume of sales of a firm and hence has an influence on its cash flow. Therefore, the weather derivative can be seen as a financial instrument to hedge primarily the volume risk associated with a product and not the price risk, whereas the commodity derivative market provides a platform to the companies to hedge price risk. However, to ensure a perfect hedge, firms enter into contracts to hedge both volume risk through weather derivative market and price risk through normal commodity market. This practice is entitled as cross-hedging.

The growth in the market of weather derivatives has been mainly influenced by the liberalization of the energy markets. For long the energy sector firms have witnessed a positive correlation between the energy prices and the weather. Weather-sensitive commodity futures that are available in the market are useful to hedge the price risk of the firm but have insignificant hedging effect on the volume risk associated with the business. In these circumstances, inventory is not a viable option to hedge this particular type of risk which is influenced by unfavourable weather conditions. The weather derivatives are hence created to address this issue related to volume risk. Furthermore, due to the high correlation of local environmental conditions and the underlying variable of the weather derivatives, the weather derivatives can provide a more effective way of risk management than already existing traditional contracts. Hence, in order to hedge the risk against bad weather conditions these firms trade in weather derivatives. The other major factor is the diminishing gap between the insurance market and the capital market. The ever-increasing growth in the issuance of catastrophic bond introduced by Chicago Board of Trade (CBOT) explains the escalating awareness of the importance bad weather affecting a firm's profit. These weather derivatives enable the companies to hedge their cash-flow against weather risk.

Pricing of traditional derivatives are based on no-arbitrage arguments, for example option pricing theory also known as the Black-Scholes pricing model. These theories consider the underlying asset of the contract to be tradable. This is one of the fundamental assumptions made in formulating the option pricing of usual financial contingent claims, based on the view of constant hedging. This assumption stands unrealistic and is violated in the case of derivatives where the underlying asset is a weather measure. Therefore, unlike traditional derivative contract, no risk-free portfolios can be created in the case of weather derivatives. Geman (1999) highlights the various issues in pricing these types of contingent claims. Hence, pricing of these types of contracts remains an issue yet to be solved.

The option pricing theory fails to acknowledge the issues related to these degree-days contracts in various fronts.

First, the payout of degree-day contract is equal to the summation of the contract value over its period. It is similar to the payoff of an Asian option. Second, the underlying asset of the weather derivative corresponds to a mean-reverting process. The weather fluctuates within a defined space over a long period, and hence it can be forecasted in short run and tends to be random around its historical mean in the long run. On the contrary the underlying asset of traditional derivatives, stock price, is stochastic in nature. Moreover, unlike conventional Black-Scholes options many degree-day derivatives have a cap to their maximum payoff. There are two primary alternative methodologies in which we can price these new set of degree-day derivatives: a) Burn Analysis which is generally used in insurance sector and b) by formulating a model which will forecast temperature - 'Temperature Model'. Hence it is more sophisticated and accurate.

The main motive of this study is to formulate a pricing model for the weather derivatives employing the 'Temperature Model'. The underlying variables for modelling weather derivatives can be temperature, daily precipitation, humidity or snowfall. The pricing model in this study considers temperature as the underlying variable because of its widespread usage and availability.

The rest of the dissertation unfolds as follows: Section 2 not only discusses the current state of the weather derivatives market describing how it has altered in the recent past, but also puts forward its advantages over the insurance contracts. It also introduces the basic terminologies related to the modelling and the pricing of these contracts. Section 3 prides a literature review and deals with the various methodologies adopted by different authors in the past to model surface air temperature and price weather derivatives. Section 4 provides a brief discussion of the data set used in this study. Section 5 provides a detail description of the modelling of temperature and the pricing of these derivative contracts. The study evaluates an arbitrage-free option pricing using a Gaussian Ornstein-Uhlenbeck model. Section 6 presents the results of the empirical analysis based on the formulated models and also based on Monte Carlo simulation. Section 7 provides examples of real time risk management applications based on these weather derivatives to hedge weather risk and finally Section 8 concludes the study.

2. An Overview of the Weather Derivatives Market

The major participant of weather derivatives market is the US compromising of over 6,000 deals with a value exceeding $45.2 billion. There are only nine cities in Europe which records for these derivatives and it is estimated that nearly 15% of the global economy is affect by unfavourable weather conditions.

In 1997 US experienced the emergence of weather derivative. This can be mainly attributed to seven factors. Firstly, during the 1990s the financial market witnessed the intertwining of the insurance and the capital markets. Previously, the insurance market defined itself separate from the capital market, until the firms started trading insurance products after insuring themselves in the capital markets. These two markets converged predominantly leading to hedging of risk in alternative markets. Considine (1999) describes weather derivatives as a rational annex of this process.

Secondly, in the year 1997 the insurance industry suffered from low premiums and consequently released considerable amount to hedge risk related to bad weather through risk capital (Considine, 1999). The occurrence of El Nino event in 1997-1998 is the third factor which triggered weather to become a derivative in 1997. The energy companies in the northern US suffered from reduced profits due to the prevalence of warm weather and the reduced consumption of gas and electricity. Fourthly, as highlighted by Clemmons 2002, the deregulation process of the electricity market which started in 1996 gave rise to competitive markets leading to electricity trading and increasing the estimation of the effect of natural calamity in both short-term and long-term demand.

The fifth factor was the increasing awareness of the energy companies to device a mechanism to mitigate this risk. Amongst various energy firms Enron was noteworthy in investigating its own revenue fluctuations due to decrease in the amount of gas pumped because to the warm winter. They in 1996 first pioneered the idea of crafting a financial instrument around an index recognizable to all energy companies, the degree-days. Unlike other financial instruments the underlying variable of this index was weather which could be independently monitored and hence the creation of this index was easy. A market was also possible as investors have varied opinions on weather fluctuations. In 1997, three US energy companies signed the first major deal.

The sixth factor became the creation of a market where these derivatives can be traded. Finally, the emerging environmental markets in 1990s particularly in the field of air pollutants and global climate change made the rise of weather derivatives market indispensable.

The emergence of the weather derivatives market thus evolved from the varied weather conditions during 1996-1997, the convergence of different financial markets and increasing awareness of environmental speculations of the various energy firms. Today the weather derivatives market has presence globally. In 1998 Scottish Hydropower was a part of the government's initiative ness to privatization and deregulation of UK energy markets. During the same year Enron sold its first weather derivative deal in UK to Scottish Hydropower. Soon the next weather derivatives deals in Europe were sold in Germany and Netherlands followed by Scandinavia. Australia and Japan also witnessed the growth of a weather derivatives market. In 2003 the World Banks conducted a survey in India to evaluate the prospect of weather derivatives in rural agricultural markets. The Chicago Mercantile Exchange (CME) in 1999 first listed the weather contract as the US market started accepting the new financial product and eventually derivative market became more open. Successively the market has grown many folds since 1997 and has diversified further into retail and construction sectors. Furthermore, the initiation of electronic trading on CME has increased the liquidity of the weather derivative market by adding in new participants. It has fuelled the weather derivatives market primarily due to four reasons:

I. It allows investors to enter contracts with no restriction on transaction size.
II. The quotes of the contract are available to all and hence encouraging participation from traders.
III. The credit risk of the investors is not accounted and hence encourages more trade participation.
IV. The electronic trading system ensures low trading cost.

Weather derivatives have fuelled companies to privatize meteorological knowledge leading to profit making. Moreover the concept of commoditization of the underlying variable in weather derivatives has initiated the process of patenting meteorological knowledge which may lead to comprehensive outcomes in future.

2.1 *The Weather Derivative Contract*

The weather derivatives are structured depending on the varying underlying weather indices. The regularly used indices are precipitation, snowfall and degree-days. Based on the above indices the weather derivatives are traded as swaps, options (call or put) and futures.

This study deals with degree-days, because of their extensive usage. Some essential definitions and terminologies governing the degree-days indices are defined below.

Definition: Temperature (in degree Celsius)
The temperature for a particular day (T_n) as recorded from a weather station is defined as:

$$T_n = \frac{T_n^{\max} + T_n^{\min}}{2} \qquad (2.1)$$

Definition: Degree-days
Heating degree-days (HDD_n) for a particular day having temperature T_n is defined as:

$$HDD_n = \max(18 - T_n, 0) \qquad (2.2)$$

Cooling degree-days (CDD_n) for a particular day having temperature T_n is defined as:

$$CDD_n = \max(T_n - 18, 0) \qquad (2.3)$$

The heating and cooling degree-days terminology was coined in US where people are more likely to heat their homes if temperature dips below 18^0 C and use air conditioners if it rises above 18^0 C. Therefore, the reference temperature level set by US is 18^0 C (65^0 Fahrenheit) which has become a standard norm.

The weather derivatives are formulated based on accumulated heating degree-days (HDDs) and cooling degree-days (CDDs). These HDDs and CDDs are clustered over a season (summer or winter) or a particular calendar month.

The total number of HDDs over a period is defined as:

$$H_N = \sum_{n=1}^{N} HDD_n \qquad (2.4)$$

The total number of CDDs over a period is defined as:

$$C_N = \sum_{n=1}^{N} CDD_n \qquad (2.5)$$

The winter months (November to March) constitutes the HDD season and the summer months (May to September) forms the CDD season. The months April and October are termed as the “shoulder months”.

2.2 *Chicago Mercantile Exchange (CME) contracts*

The CME Degree-day Index compromises of cumulative HDDs and CDDs. Futures on this index are traded in CME. Presently there are in total thirty five cities listed to trade on monthly and seasonal weather index futures and options on futures. These include eighteen US cities, nine European cities, six Canadian cities and two in Japan.

The Degree-days Index futures are agreements, permitting the buying and selling of HDD/CDD index value at a specified future date. The size of the trading unit is 20 British pound times the CME European degree-days index. The futures contracts are daily market-to-market and lead to cash settlement in the investor’s account.

A CME degree-days call option gives the owner of the contract the right to purchase a HDD/CDD futures contract at a pre-stated exercise (strike) price. Similar, CME degree-days put option gives the owner of the contract the right to sell a HDD/CDD futures contract at a specified exercise (strike) price. The position limit of a monthly futures call option is 10,000 contracts. These futures are of European style allowing the contract to be exercised only at the expiration date.

2.3 *Weather Options*

Apart from the CME, put and call options are also traded on the over-the-counter (OTC) market. At the starting of the contract the buyer of the degree-days call option pays a premium to the seller of the option. The buyer receives a payout if at the expiration of the contract the number of degree-days exceeds the specified strike level. The amount received by the option-holder for the period when the degree-days exceed the strike level is the tick size of the contract. The final payoff received by the holder is determined based on the strike and the tick size of the contract. Unlike regular stock options these weather options often have a limit (cap) on the maximum payoff.

A weather option can be modelled based on the following defined parameters:

I. The type of the contract (call or put options)
II. The period of the contract (N days)
III. The underlying degree-days index – HDD or CDD
IV. The strike level (K)
V. The tick size (f)
VI. The cap on maximum payout
VII. Authenticated temperature data from a official weather station

Computing the value of accumulated HDDs from equation (2.4), the payoff of a HDD call option (uncapped) can be defined as:

$$\prod_{c} = \varphi \max(H_N - K, 0) \tag{2.6}$$

Similarly, a HDD put option can be written as:

$$\prod_{p} = \varphi \max(K - H_N, 0) \tag{2.7}$$

The payoffs of CDD call and put options can be written as follows: $\prod_{c} = \varphi \max(C_N - K, 0)$ and

$$\prod_{p} = \varphi \max(K - C_N, 0) \tag{2.8}$$

2.4 Weather swaps

A swap is an agreement between two parties, which interchanges risk for a specified period of time. In this type of contract one party pays a fixed price whereas the other side makes a floating payment.

The weather risk swaps enables the two parties to exchange specific exposures to weather risk for a predetermined time period. These are over the counter financial products, which can be customized to meet the needs of both the parties signing the contract. The weather swaps can often be regarded as a forward contract as the parties in the contract swap their cash-flows in a particular day. Most of these swaps have a single calendar month period.

2.5 The difference: Weather derivatives and insurance contracts

The payoff of an insurance contract is disbursed to the holder if he is able to provide evidence of his suffered financial losses. Unlike these insurance contracts, in weather derivatives the payouts are based on actual observable weather outcomes, regardless of its influence on the holder of the instrument. To reap the benefits of a weather derivative an investor need not indulge in weather sensitive production.

The insurance contracts are mostly structured to protect the holder from natural calamities like earthquakes and cyclones having very low probability of occurrence and not from regular weather uncertainties. On the contrary, weather derivatives can be designed to have payoffs under any weather conditions. These weather derivatives prove to be highly beneficial for the companies which are closely affected by uncertain regular weather conditions and can cause huge losses. Moreover, weather derivatives enable one investor to book profits in warm winters whereas another investor benefits from very cold winters. The derivates market allows these investors to enter into agreement and hedge each other’s weather risk, which is not possible in insurance markets. This further lowers the cost of hedging risk and increases the liquidity in the market. Additionally, unlike insurance, investors can enter into weather derivatives contract for speculative purposes.

3. Literature Review

This section highlights the previous work done in the field of modelling price for weather derivatives. This section aims to understand and analyze the prior work done and form a basis for accurate modelling of temperature, the underlying asset of weather derivative, for the next section. The weather derivatives are generally traded much before the commencement of the contract period. Hence, it is very crucial to have an accurate estimate of the temperature. This emphasizes the importance of a precise temperature forecasting model.

Cao and Wei (2000) and Caballero, Jewson and Brix (2002) carried out studies which considered temperature behaviour as a time series data to explain its variations and properties. The studies carried out can be primary classified under two categories in which the authors try to explain the regular temperature variance. These two approaches include: a) forecasting and fitting of the temperature variations through a time-series approach and b) using a financial diffusion process to estimate the temperature movements. The other authors who have incorporated the first procedure include Campbell and Diebold (2002), and Caballero and Jewson (2003). Whereas Dischel (1998), Brody, Syroka and Zervos (2002) carried out studies in this area fitting temperature in a financial diffusion process. Furthermore similar sort of studies were carried out by Alton, Djehiche and Stillberger (2002) and Benth and Saltyte-Benth (2005). But, majority of the studies carried out in this aspect of weather derivatives excludes the seasonal effect of the temperature pattern from the mean and the deviation dynamics of temperature movements.

3.1 A Time-series approach

Modelling of temperature to forecast daily temperature using a time-series approach follows a discrete process. This approach is acceptable under the autoregressive property of the temperature pattern. It has been considerable noticed over long time horizon that the temperature on a particular day shows dependency characteristics on the temperature of the previous day. More importantly, the values of HDD and CDD indices are calculated using discrete daily temperature values and hence following a discrete temperature forecasting approach makes it easier. The main drawback of using a continuous process is because of its Markovian[1] characteristics. All the authors mentioned above who have carried out their study based on this discrete process have forecasted temperature based on a class of Autoregressive Moving-Average (ARMA) models.

Cao and Wei (2000) modelled the daily temperature movements without the consideration of mean and the pattern trend. They represented the temperature ($U_{yr,t}$) as:

$$U_{yr,t} = Y_{yr,t} - \hat{\hat{Y}}_{yr,t} \quad (3.1)$$

$$\hat{\hat{Y}}_{yr,t} = \frac{1}{N}\sum_{yr=1}^{N} T_{yr,t} \quad (3.2)$$

Where $Y_{yr,t}$ defines the temperature of a particular date of a particular year. $\hat{\hat{Y}}_{yr,t}$: denotes the historical mean temperature. The authors proposed a model with periodic variance having a k-lag autocorrelation structure represented as:

$$U_{yr,t} = \sum_{k=1}^{K} \rho_i U_{yr,t} - k + \rho_{yr,t} \times \xi_{yr,t} \quad (3.3)$$

$$\sigma_{yr,t} = \sigma - \sigma_1 \mid \sin\left(\frac{\pi t}{365} + \phi\right) \mid \quad (3.4)$$

The study used data for five US cities for a period of twenty years. They modelled the daily volatility of the temperature ($s_{yr,t}$) using a sinusoidal function. This captures the seasonal behaviour of the temperature. The random source in equation (3.3) has a distribution defined by: $\xi_{yr,t} \sim iidN(0,1)$.

Cambell and Diebold (2002) later extended the autoregressive structure by the inclusion of a Fourier series[2] in both the mean and the variance models in equations (3.3) and (3.4). The authors represented the equations as:

$$T_t = \beta_0 + \beta_1 + \sum_{p=1}^{P}\left(\delta_{c,p}\cos\left(\frac{2\pi pt}{365}\right) + \delta_{s,p}\left(\frac{2\pi pt}{365}\right)\right) + \sum_{k=1}^{K}\rho_k T_{yr,t-k} + \sigma_{yr,t} \times \varepsilon_{yr,t} \quad (3.5)$$

$$\sigma^2{}_t = \sum_{q=1}^{Q}\left(\gamma_{c,q}\cos\left(\frac{2\pi qt}{365}\right) + \gamma_{c,q}\sin\left(\frac{2\pi qt}{365}\right)\right) + \sum_{r=1}^{R}\alpha_r \times \varepsilon^2{}_{t-r} \quad (3.6)$$

$$\varepsilon \sim iid(0,1) \quad (3.7)$$

These equations revealed a better forecast for the temperature movements and captured the seasonal movements as compared to those proposed in the equations developed by Cao and Wei (2000). The major problem in their model was the equation (3.2) used for the calculation of the historical average. Moreover, the addition of a proxy for Fourier series in the above equations renders the model very economical by decreasing the number or estimators as compared to other temperature models capturing the seasonality effect. Cambell and Diebold (2002) data set constituted ten U.S. stations over a period of over forty years, 1st January 1960 to 11th May 2001. Hence, it is much likely that the linear trend used to capture the increase in temperature year on year will be underestimated.

Caballero, Jewson and Brix (2002) showed that the autoregressive models failed to capture the persistent auto-correlation characteristic of temperature variations. Hence, these models led to considerable underestimation of pricing of weather derivatives. The authors in their study estimated an Auto-regressive Fractionally Integrated Moving Average (ARIFMA) model to resolve the above mentioned issue. Caballero, Jewson and Brix (2002) implemented maximum likelihood estimation (MLE) method to their data set which constituted daily temperature record for two hundred and twenty two years for Central England and for fifty years for Chicago and Los Angles. However, the use of MLE and the fitting of ARIFMA model over a wide data range is a tedious process.

Through empirical observations Caballero and Jewson (2003) showed that the temperature anomalies during the summer season are more significant than the winters. They hence showed that the ARIFMA model failed to reflect the seasonality in the temperature model, represented as an autoregressive function. They developed an Autoregressive on Moving Average (AROMA) model to estimate a particular day's temperature as the summation of temperature movements over different time-scales. The model also adds a Gaussian white-noise parameter. They further extend their AROMA model to resolve the issue of seasonality through SAROMA (Seasonal Autoregressive on Moving Average. Caballero and Jewson (2003) proposed to estimate a different model for each day of the year, having a different (a_i) value. The authors estimated their SAROMA model on Miami daily temperature data for a period of forty years. They estimated the following model:

$$\widetilde{T}_{y,t} = \sum_{i=1}^{M}\left(\alpha_i \sum_{n-1}^{n-m} \widetilde{T}_{y,t}\right) + \varepsilon_{y,t} \quad (3.8)$$

The AROMA includes (m_1, m_2……m_M) processes. In this framework, for a proper estimation of the parameters the number of moving averages should be small. The major drawback of this method is its implementation.

3.2 *A Continuous Process approach*

Many authors in the prior studies have applied a continuous process to define daily temperature. These

[1] A Markov process is a random process and only considers autocorrelation having one lag.
[2] A Fourier series breaks down complex periodic functions as a linear combination of simple oscillating sine or cosine functions.

approaches are quite similar to those applied to describe the short-term interest rate. The framework is a financial diffusion process and is closely related to the interest-rate derivative pricing approach defined by Hull and White (1990). Assuming the daily temperature as T (t), the speed of mean reversion as a (t) and the long-term level as K (t), the continuous process for weather derivatives is defined as:

$$dT(t) = a(t)[K(t) - T(t)]dt + \sigma(t)dW(t) \qquad (3.9)$$

The term dW(t) captures the random shocks of the process and s (t) is the temperature movement volatility.

The process as represented in the equation (3.9) is time dependent, therefore the speed of mean reversion and the volatility are independent of temperature and are a function of time. This framework captures the seasonal effect and the increasing temperature trend as a mean-reversion process which is time-dependent rather than having a specific value. These models outperform the discrete time model as they fail to include the effect of natural continuous-time pattern of temperature variations. Within this class of approach Dischel (1998) and Alaton et al. (2002) assumed dW (t) as a Gaussion distribution, whereas Brody et al. (2002) considered a fractional Brownian motion to include the dependency characteristics of temperature in the long-run.

3.2.1 Gaussian Distributions

Dischel (1998) in his work pioneered a two parameter model for describing a continuous process approach. In his model he considered the daily temperature and its distribution separately. Considering a (t) constant and K (t) as the average of day-to-day temperature as calculated from the equation (3.2), the random part of the equation (3.9) is stated as:

$$\sigma t W t = \lambda dw_1(t) + \beta dw_2(t) \qquad (3.10)$$

In the above equation dw_1 (t) and dw_2 (t) are two separate Wiener processes defining the time-variant temperature drive T (t) and the temperature variation DT (t), respectively.

Dornier and Queruel (2000) argued that the mean-reverting model proposed by Dischel (1998) actually failed to revert to its specified mean in a long-horizon time frame. They added an extra variable in the right hand side of the equation (3.9) to make the model mean-reverting. Their model adds the seasonal variations proxy and it takes the form:

$$dT(t) = dK(t) + a(t)[K(t) - T(t)]dt + \sigma(t)dW(t) \qquad (3.11)$$

Alaton et al. (2002) modified the mean-reverting model proposed by Dischel (1998) and also considered the changes proposed by Dornier and Queruel (2000). Their model further incorporates the linear trend of year-on-year increase in surface temperature. Unlike the models proposed by Dischel (1998) Torro et al. (2003) did not incorporate the factor dK(t) and hence their equation failed to capture the mean-reversion aspect of temperature in the long-run. Their study consisted of temperature data for twenty nine years for four Spanish weather stations.

Brody et al. (2002) formulated their model for a sample period, 1772 to 1999, of daily England temperature. They rejected the use of Brownian process in the equation (3.9) as it failed to explain the long-run persistent correlation in temperature variations. Similar to Cabellero et al. (2002), who used ARIFMA models to solve the issue by de-trending the temperature movements, included a Fractional Brownian motion. One of the major issues aligned with this model is its implementation as it fails to be a Markov or a semi-martingale process.

3.2.2 Non-Normal Distributions

Apart from Dischel (1998) all the other studies assume Gaussian distribution of temperature variation. The work carried by Dischel (1998) did not assume any particular pattern of distribution of temperature movements.

The non-Gaussian distribution of the data can be dealt in a number of ways to model temperature movements. Benth and Saltyte-Benth (2005) used generalized hyperbolic Levy noise to deal with the skewed temperature fluctuations data involving heavy tails. Brody et al. (2002) introduced a simple harmonic motion to address the drift in the seasonality of temperature variations.

4. Data Description

The data used for modelling temperature is taken for four cities in UK namely London, Bristol, Louisville and Nashville. The data set consists of daily temperature data for the last 30 years from 1st July 1978 to 1st July 2008. The time-series weather data consists of maximum and minimum temperatures measured in Fahrenheit degrees and the degree-days indices. The data set are obtained from CME data base and from Kentucky Climate Data base. The sample data for each of the weather stations consists of 10,959 observations each.

The following table reports the descriptive statistics of the daily average temperature data over the last 30 years.

I - Table 4.1: Descriptive Statistics

	London (daily Average)	*Nashville(daily Average)*	*Louisville(daily Average)*	*Bristol(Daily Average)*
Mean	13.657	15.509	15.075	14.496
Standard Error	0.088	0.089	0.102	0.091
Median	14.722	16.389	16.200	15.600
Mode	23.056	25.833	27.000	25.200

Standard Deviation	9.180	9.368	10.726	9.501
Sample Variance	84.267	87.760	115.054	90.274
Kurtosis	-0.667	-0.706	-0.744	-0.744
Skew-ness	-0.435	-0.410	-0.381	-0.394
Range	51.389	53.889	61.200	56.400
Minimum	-21.111	-20.278	-24.300	-24.900
Maximum	30.278	33.611	36.900	31.500
Count	10959	10959	10959	10959

The descriptive statistics shows high values of standard deviation across all the stations, revealing that the trajectory of the time-series temperature data oscillates over time. The mean and the standard deviation of the daily average weather temperature-data vary across the stations because of its geographical location. The value of skew-ness and kurtosis is significantly different from zero, stating that the data is not symmetrically distributed. The data set exhibits negative kurtosis, platykurtic[3]. The normality test statistic values significantly reject the null hypothesis of normality for all the four cities. The temperature data has a bimodal pattern having peaks for summer and winter seasons respectively, as shown in figure (5.2). The normality test values are tabled below:

II - Table 4.2: Normality Test Values

Tests	London	Louisville	Nashville	Bristol
Asymptotic	548.59	517.90	535.32	536.68
	[0.0000] **	[0.0000] **	[0.0000] **	[0.0000] **
Normality	1231.40	1110.90	1180.50	1174.30
	[0.0000] **	[0.0000] **	[0.0000] **	[0.0000] **

(** - 5% significant level)

4.1 Unit-Root Testing

The aim of this analysis is to test the seasonality-frequency cycle of the weather data. The null hypothesis stated here is that the daily average temperature approximates a covariance-stationary process. The presence of unit root is tested to determine whether the process is stationary, which is important to avoid spurious results. The table below reports the Augmented-Dickey Fuller test results.

III - Table 4.3: Unit-Root Test Values

	London	Louisville	Nashville	Bristol
ADF	-8.454	-8.022	-8.102	-7.889
Statistic (AIC Value)	(2.368)	(2.548)	(2.348)	(2.274)

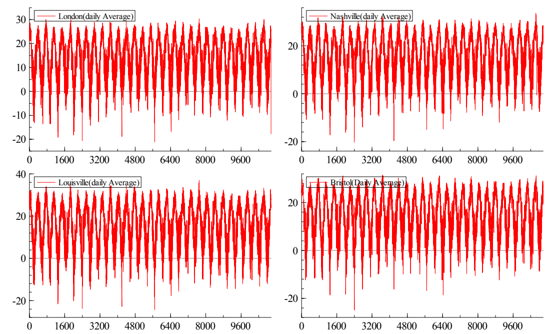
The Augmented-Dickey Fuller test is performed under a lag length, which is selected by minimizing Akaike-Schwartz criteria under controlled Durbin Watson serial correlation test. The critical value of the test at 5% confidence level is -2.8622. The test results reveal that the time-series data shows no evidence of non-stationary. Therefore, the application of Fourier series will be appropriate on the time series data.

5. Methodology

5.1 Modelling Temperature

This section formulates a model to captures the behaviour of daily temperature, which is the underlying variable of weather derivatives. While pricing weather derivatives it is necessary to know the process in which the weather varies. The primary aim is to model the temperature movements as a stochastic process. The Following figure shows the mean temperature of the four cities:

I - Figure 5.1: Mean Temperature Plot



5.1.1 Seasonal Variation

The above figure illustrates the effect of seasonality in temperature behaviour. The mean temperature at London station varies from 23^0C in the summers to $(-4)^0$C in the winters. The figure shows that the weather seasonality traces a sine-function. The trajectory can be formulated, by defining time (t), the oscillating seasonal frequency (ω) and a phase difference (Ø) as:

$$\sin(2\pi\omega t + \phi) \quad (5.1)$$

In the above sine-function the oscillation frequency is one year (leap year not considered), therefore we

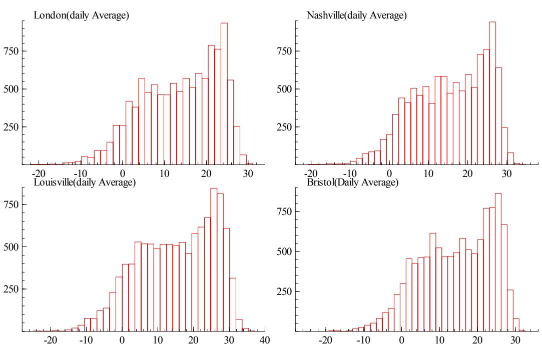
[3] The data set has a smaller peak at its mean and exhibits thin tails.

take $\omega = \left(\frac{1}{365}\right)$[4]. A phase difference (Ø) is introduced in the equation as 1st July and 1st January does not always record for the maximum and minimum temperature respectively.

5.1.2 Temperature Distribution

The temperature distributions of the four cities (London, Bristol, Louisville and Nashville) are shown in the following histogram:

II - Figure 5.2: Temperature Distribution Plot



The figure shows a bimodal distribution of temperature time series weather data and also reveals that there is a weak upward trend in the mean recorded temperature. The reasons which can support this are (1) the rise in world temperature due to global warming and (2) the urban-heating effect[5]. Therefore the modelling of the temperature is carried out in two phases.

I. The first phase in modelling the temperature process is based on the deterministic part. It concerns with defining both the long-term and seasonal trends of the time-series data and from there,
II. A stochastic model is fitted to the residuals. Hence, the overall temperature process is fitted into a Stochastic Differential Equation (SDE).

The positive trend in the rise of mean temperature is assumed to be quadratic[6]. This approach is taken to de-trend the data and reveal a more accurate approximation of the temperature pattern. Therefore, the mean temperature at a particular time (t) can be represented as:

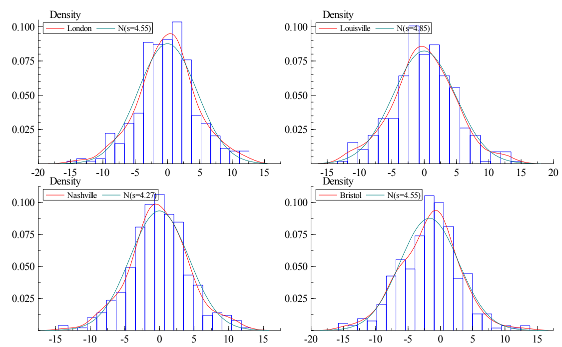
$$\overline{T}_t = B_0 + B_1 t + B_2 t^2 + \alpha \sin(2\pi\omega t + \phi) + \mathrm{E}_t \qquad (5.2)$$

The parameters B_0, B_1, and a, amplitude of the sine function, are estimated to fit the weather data trajectory. The 'E_t' represents the noise process, as the temperature movements are nondeterministic. This model aims in capturing the seasonal effect better than the one proposed by Alaton et al. (2002).

5.1.3 The noise process

The following figure shows the daily temperature differences, residuals, plotted against density.

III - Figure 5.3: Density Plot of Daily Temperature Differences



The noise process (E_t) is considered as a stochastic process, Wiener process, with drift (m) and variance (s^2) defined by:

$$E_t = \mu_t t + \sigma_t W_t \qquad (5.3)$$

The time series data highlights a temperature variance across the months, whereas reveals no significant drift in value within each month of a particular year. Therefore, the noise process (E_t) can be estimated as a Wiener process $(\sigma_t W_t, t \geq 0)$. The variance s_t is a positive constant for each month of a year. It is defined as $\sigma_t = [\sigma_i]_{i=1}^{12}$, where the variable 'i' denotes the 12 months, January to December.

5.1.4 The mean-reverting process

The temperature rises and falls for a short period and then reverts back to its mean. Hence, the stochastic process capturing the movements of temperature follows a mean-reverting process. The model of the temperature following an Ornstein-Uhlenbeck process[7] has the following stochastic differential equation (SDE):

[4] The value of 'ωt' is taken as $(2\pi/365)$ to remove the impact of Dopplers effect in the long-term projection of temperature.
[5] Urban-Heating effect reflects to the rise in temperature near big urban cities because of its development and having a warming effect on its surrounding areas.
[6] The warming effect is weak and hence in case of a higher polynomial approximation only the linear and the quadratic terms will define its effect.

[7] Ornstein-Uhlenbeck process: It is a Gaussian process having a stationary probability distribution. It has a bounded variance.

$$dT_t = -\theta\left(T_t - \overline{T}_t\right)dt + \sigma_t dW_t \tag{5.4}$$

Here $\theta \in R$ is an estimate for the speed of mean-reversion. The stationary variance of the above defined mean-reverting process is: $\operatorname{var}(T_t) = \frac{\sigma_t^2}{2\theta}$. Dornier and Queruel (2002) stated that the SDE (5.4) fails to capture the actual mean-reversion process as the mean temperature $\left(\overline{T}_t\right)$ is not a constant term and varies with time. Hence, a time derivative of $\left(\overline{T}_t\right)$ is included in the drift component of the SDE. The added term is:

$$\frac{d\overline{T}_t}{dt} = B_1 + 2B_2 t + 2\pi\omega\alpha\cos(2\pi\omega t + \phi) \tag{5.5}$$

Now, the mean-reverting SDE satisfying the condition $\mathrm{E}[T_t] = \overline{T}_t$ is given by:

$$dT_t = \left[\frac{d\overline{T}_t}{dt} - \theta\left(T_t - \overline{T}_t\right)\right]dt + \sigma_t dW_t \tag{5.6}$$

$$df(T_t, t) = e^{\theta t} dT_t + \theta T_t e^{\theta t} dt$$

$$= \left[\left\{\frac{d\overline{T}_t}{dt} + \theta\left(\overline{T}_t - T_t\right)\right\} e^{\theta t}\right]dt + \sigma e^{\theta t} dW_t$$

Taking $f(T_t, t) = T_t e^{\theta t}$ and applying Ito's Lemma we get:

(5.7)

Integrating the above equation (5.7) from t_0 to t we get:

$$\int_{t_0}^{t} df = \int_{t_0}^{t}\left[\left\{\frac{d\overline{T}_t}{dt} + \theta\left(\overline{T}_t - T_t\right)\right\} e^{\theta t}\right]dt + \int \sigma_t e^{\theta t} dW_t$$

$$\Rightarrow T_t e^{\theta t} = T_{t_0} + \int_{t_0}^{t} d\overline{T}_t e^{\theta t} + \int_{t_0}^{t} \theta\left(\overline{T}_t - T_t\right) e^{\theta t} dt + \int_{t_0}^{t} \sigma_t e^{\theta t} dW_t$$

$$\Rightarrow T_t = \overline{T}_t + \left(T_{t_0} - \overline{T}_{t_0}\right) e^{-\theta(t - t_0)} + \int_{t_0}^{t} e^{-\theta(t-\tau)} \sigma_\tau dW_\tau$$

Therefore, the model of temperature can be represented as:

$$T_t = \overline{T}_t + \left(T_{t_0} - \overline{T}_{t_0}\right) e^{-\theta(t - t_0)} + \int_{t_0}^{t} e^{-\theta(t-\tau)} \sigma_\tau dW_\tau \tag{5.8}$$

5.1.5 Estimation of Parameters

The long-term upward trend in the temperature is estimated by a quadratic model. The quadratic model is therefore defined as:

$$T_{Quadratic} = \alpha_0 + \alpha_1 t + \alpha_2 t^2 \tag{5.9}$$

For the purpose of parameter analysis the seasonal variation of temperature is defined as a truncated Fourier series, represented as:

$$T_{Seasonal} = k + \alpha_3 \sin(2\pi\omega t) + \alpha_4 \cos(2\pi\omega t) \tag{5.10}$$

The constant parameter (k) in equation (5.10) is replaced by the quadratic trend to form the temperature estimation equation of the form:

$$\hat{\overline{T}}_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 \sin(2\pi\omega t) + \alpha_4 \cos(2\pi\omega t) \tag{5.11}$$

A least-square estimation is performed on the above equation to find the values of the parameters. The values of the parameter constants in (5.2) can be calculated as:

$$B_0 = \alpha_0 \tag{5.12}$$

$$B_1 = \alpha_1; B_2 = \alpha_2 \tag{5.13}$$

$$\alpha = \sqrt{\left(\alpha_3^2 + \alpha_4^2\right)} \tag{5.14}$$

$$\phi = \tan^{-1}\left[\left(\frac{\alpha_4}{\alpha_3}\right)\right] - \pi \tag{5.15}$$

The mean temperature function based on the above model for all the four cities are estimated in the later section (6) of the study.

5.1.6 Estimation of noise variance (σ)

Defining a particular month as 't', the number of days in that month as 'N_t' and the recorded temperatures for that month as T_i, i=1 to N_m, the value of s can be estimated as:

$$\hat{\sigma}_t^2 = \frac{1}{N_t}\sum_{i=1}^{N_t}\left(T_i - T_{i-1}\right)^2 \tag{5.16}$$

The above estimation is attributed to the quadratic variation of daily temperature. Basawa and Rao (1980) in their study estimated the variance in the same procedure as per equation (5.16). The table (6.1) shows the estimated value of s based on the above estimation technique.

5.1.7 Estimation of the speed of mean reversion (θ)

An autocorrelation coefficient (k) is initially calculated by regressing today's temperature (T_i) against yesterday's temperature (T_{i-1}) and then fitted in to the following equation to estimate the value of mean-reversion.

$$\theta = e^{-k} \tag{5.17}$$

Table (6.2) shows the estimated value of mean-reversion based on the above estimation equation.

5.2 Pricing Weather Derivatives

Temperature, the underlying variable of the weather derivative, is not tradable and hence the risk premium estimator (b) is considered to find the price of a weather derivative contract. The two assumptions made apart from considering a risk-free asset of interest rate 'r' to price the weather derivatives in this study are: (1) the risk premium factor is considered constant as there is no authentic market to get the prices and (2) the payoff of the contract is a single unit of currency for every unit of degree Celsius.

In a risk neutral environment under martingale measure (Q), taking Girsanov's transformation the wiener process (W_t) in the temperature model represented in equation (5.4) can be defined as:

$$\widetilde{W}_t = Wt + \beta t \tag{5.18}$$

By differentiating the above equation and rearranging we get:

$$d\widetilde{W}_t = dW_t + \beta dt \Rightarrow dW_t = d\widetilde{W}_t - \beta dt \tag{5.19}$$

Putting the value in equation (5.6) we get:

$$dT_t = \left[\frac{d\overline{T}_t}{dt} - \theta\left(T_t - \overline{T}_t\right)\right]dt + \sigma_t\left(d\widetilde{W}_t - \beta dt\right) \tag{5.20}$$

Since, the price C_t follows the same process as temperature (T_t). Rearranging the above equation the price (C_t) follows the underlining process, under a martingale measure (Q):

$$dC_t = \left[\frac{d\overline{T}_t}{dt} - \theta\left(T_t - \overline{T}_t\right) - \beta dt\right]dt + \sigma_t d\widetilde{W}_t \tag{5.21}$$

The price of the derivative contract for all times $t \le T$ is Q-martingale and is expressed as a discounted expected value by:

$$C_t = E_Q\left(C_T \mid F_t\right) \tag{5.22}$$

In terms of physical measure (P), taking Radon-Nikodym derivative of Q the above equation can be written as:

$$C_0 = E_P\left(\frac{dQ}{dP}C_T\right) \tag{5.23}$$

Since in both the equations (5.8) and (5.21) only the drift term is different, the variance of both the T_t and C_t is the same. Therefore, the variance of C_t is equal to:

$$\operatorname{var}\left[C_t \mid F_{t_0}\right] = \int_{t_0}^{t} e^{-2\theta(t-\tau)}\sigma_\tau^{\,2}\,d\tau \tag{5.24}$$

Solving the above equation we get:

$$\operatorname{var}\left[C_t \mid F_{t_0}\right] = \frac{\sigma_i^{\,2}}{2\theta}\left(1 - e^{-2\theta(t-t_0)}\right) \tag{5.25}$$

Since the temperature and the price follow the same Q-martingale process, the covariance of temperature at interval $0 \le t_0 \le t \le t_1$ is defined as:

$$\operatorname{cov}\left(T_t, T_{t_1}\right) = e^{-\theta(t_1 - t)}\operatorname{var}\left[T_t \mid F_{t_0}\right] \tag{5.26}$$

Further, from equation (5.21) we get:

$$E_Q\left[C_t \mid F_{t_0}\right] = \overline{T}_t + \left(T_{t_0} - \overline{T}_{t_0}\right)e^{-\theta(t-t_0)} + \int_{t_0}^{t}\beta e^{-\theta(t-\tau)}\sigma_\tau d\tau \tag{5.27}$$

$$\Rightarrow E_Q\left[C_t \mid F_{t_0}\right] = \overline{T}_t + \left(T_{t_0} - \overline{T}_{t_0}\right)e^{-\theta(t-t_0)} - \frac{\beta\sigma_i}{\theta}\left(1 - e-\theta(t-t_0)\right) \tag{5.28}$$

Now, with reference to equation (5.8) we can write:

$$E_P\left[T_t \mid F_{t_0}\right] = \overline{T}_t + \left(T_{t_0} - \overline{T}_{t_0}\right)e^{-\theta(t-t_0)} \tag{5.29}$$

Putting the value of the equation in equation (5.27) we get:

$$E_Q\left[C_t \mid F_{t_0}\right] = E_P\left[T_t \mid F_{t_0}\right] - \frac{\beta\sigma_i}{\theta}\left(1 - e^{-\theta(t-t_0)}\right) \tag{5.30}$$

Following equations (5.27) and (5.29), we evaluate $E_Q\left[C_t \mid F_{t_0}\right]$ for a time frame $0 \le t_0 \le t_1 \le t \le t_m$, where the month is taken as $[t_1, t_m]$ equal to:

$$E_Q\left[C_t \mid F_{t_0}\right] = E_P\left[T_t \mid F_{t_0}\right] + \frac{\beta\sigma_i}{\theta}e^{-\theta(t-t_0)} - \frac{\beta}{\theta}\left(\sigma_i - \sigma_l\right)e^{-\theta(t-t_1)} + \frac{\beta\sigma_l}{\theta} \tag{5.31}$$

Similarly, from equation (5.24) and (5.25) we calculate the $\operatorname{var}\left[C_t \mid F_{t_0}\right]$ for a time frame $0 \le t_0 \le t_1 \le t \le t_m$, where the month is taken as $[t_1, t_m]$, as:

$$\operatorname{var}\left[C_t \mid F_{t_0}\right] = \frac{1}{2\theta}\left(\sigma_i^{\,2} - \sigma_l^{\,2}\right)e^{-2\theta(t-t_1)} - \frac{\sigma_i^{\,2}}{2\theta}e^{-2\theta(t-t_0)} + \frac{\sigma_l^{\,2}}{2\theta} \tag{5.32}$$

Also since, the price C_t follows the same process as temperature (T_t) we have:

$E_Q\left[C_t \mid F_{t_0}\right] = E_Q\left[T_t \mid F_{t_0}\right]$ and
$$E_Q\left[C_t \mid F_{t_0}\right] = E_Q\left[T_t \mid F_{t_0}\right] \quad (5.33)$$

5.2.1 Pricing of Heating-degree day (HDD) option

Temperature, as pointed out before, is the most common underlying asset used in a weather derivative option where heating-degree days (HDD) and cooling-degree days (CDD) are used to define temperature.

This section of the study models the price of a HDD option. The payoff of an Asian option is:

$$\prod_c = \varphi \max\left(H_N - K, 0\right) \quad (5.34)$$

The above contract is an arithmetic average since:

$$H_N = \sum_{n=1}^{N} HDD_n \quad (5.35)$$

For an underlying asset (temperature) which is log-normally distributed, we calculate the values of the mean (m_t) and the variance (s_t^2) as per the equations (5.31) and (5.32). Therefore, at time t_0 under Q-martingale measure we have the distribution of temperature as:

$$T_t \sim N\left(\mu_t, \sigma_t\right) \quad (5.36)$$

Given the information distribution for the underlying variable, temperature, we calibrate the price of a weather derivative contract, whose payoff is a function of accumulated HDDs during that particular time. Let us, assume that we want to calculate the price of a weather derivative contract stationed at London for the month of December. During this winter month the probability of temperature exceeding 18^0C for a particular day (n) is minimal, that is we are likely to have: ($HDD_n = \max\left(18 - T_n\right) \neq 0$). This contract can be written as:

$$H_N = 18N - \sum_{n=1}^{N} T_n \quad (5.37)$$

As stated earlier, the temperature (T_n) follows an Ornstein-Uhlenbeck process and therefore $\sum_{n=1}^{N} T_n$, which is linear summation of the elements in the vector (T_1, T_2, $T_3 \ldots T_n$) also follows an Ornstein-Uhlenbeck process. Hence, it can be stated that H_N also follows an Ornstein-Uhlenbeck process. Under Q-martingale measure the moments are calculates as:

$$E_Q\left[H_N \mid F_t\right] = E_Q\left[18N - \sum_{n=1}^{N}\left(T_n \mid F_t\right)\right]$$

$$E_Q\left[H_N \mid F_t\right] = 18N - E_Q \sum_{n=1}^{N}\left(T_n \mid F_t\right) \quad (5.38)$$

Similarly, the variance for $T_n < T_l$ is calculated as:

$$\operatorname{var}\left[H_N \mid F_t\right] = \sum_{i=n}^{N} \operatorname{var}\left[T_N \mid F_t\right] + 2\sum\sum \operatorname{cov}\left[T_n T_l \mid F_t\right] \quad (5.39)$$

5.2.2 Out-of-Period Valuation

Now, taking $\operatorname{var}[H_N|F_t]=s_N^2$ and $E_Q[H_N|F_t]=m_N$, we have $H_N \sim N(m_N, s_N)$. Therefore, the price of the call option [c(t)], under Q-measure for $t \geq t_0$ and with (r) as the risk-free rate of investment is:

$$c(t) = e^{-r(t-t_0)} E_Q\left[\max\left\{\left(H_N - K\right), 0\right\} \mid F_t\right] \quad (5.40)$$

Solving the right hand side of the above equation we get:

$$E_Q\left[\max\left\{\left(H_N - K\right), 0\right\} \mid F_t\right] = \int_K^{\infty} \left(H_N - K\right) f H_N \, dH_N \quad (5.41)$$

Defining a new standardized variable (z) for $H_N \sim N(m_N, s_N)$:

$$z = \frac{H_N - \mu_N}{\sigma_N} \quad (5.42)$$

The distribution is:

$$fz = \frac{1}{2\pi} e^{\frac{-z^2}{2}} \quad (5.43)$$

Putting the values from equation (5.42) in equation (5.41) and taking $b = \frac{K - \mu_N}{\sigma_N}$ we get:

$$\int_K^{\infty} \left(H_N - K\right) f H_N \, dH_N = \int_b^{\infty} \left(\mu_N - K + \sigma_N z\right) dz$$

$$= \left(\mu_N - K\right) \int_b^{\infty} fz dz + \sigma_N \int_b^{\infty} z f z dz$$

$$= \left(\mu_N - K\right) \psi(-b) + \frac{\sigma_N}{\sqrt{2\pi}} \int_b^{\infty} z e^{-\frac{z^2}{2}} dz$$

$$\int_K^{\infty} \left(H_N - K\right) f H_N \, dH_N = \left(\mu N - K\right) \psi(-b) + \frac{\sigma_N}{\sqrt{2\pi}} e^{\frac{-b^2}{2}} \quad (5.44)$$

Putting the value of the equation (5.44) in equation (5.33) we get the value of the HDD call option as:

$$c(t)=e^{-r(t-t_0)}\left[(\mu_N-K)\psi(-b)+\frac{\sigma_N}{\sqrt{2\pi}}e^{\frac{-b^2}{2}}\right] \quad (5.45)$$

In the above equation (5.45), Ψ is the cumulative distribution function for the standardized normal distribution.

Similarly, the value of a HDD put option p(t) is:

$$p(t)=e^{-r(t-t_0)}E_Q\left[\max\{(K-H_N),0\}\mid F_t\right] \quad (5.46)$$

Taking the right hand side of equation (5.46) and solving as per equations (5.42), equation (5.43) and putting $b=\frac{K-\mu_N}{\sigma_N}$ we get:

$$E_Q\left[\max\{(K-H_N),0\}\mid F_t\right]=\int_0^K (K-H_N)fH_N\,dH_N$$

$$=\int_{-\frac{\mu}{\sigma}}^{b}(-\sigma_N z+K-\mu_N)fzdz$$

$$=-\sigma_N\int_{-\frac{\mu}{\sigma}}^{b}zfzdz+\int_{-\frac{\mu}{\sigma}}^{b}(K-\mu_N)fzdz$$

$$=-\frac{\sigma_N}{\sqrt{2\pi}}\int_{-\frac{\mu}{\sigma}}^{b}ze^{\frac{-z^2}{2}}dz+(K-\mu_N)+(k-\mu_N)\left(\psi(b)-\psi\left(\frac{-\mu_N}{\sigma_N}\right)\right)$$

$$=\frac{\sigma_N}{\sqrt{2\pi}}\left(e^{-\frac{b2}{2}}-e^{-\frac{1}{2}\left(\frac{-\mu N}{\sigma N}\right)^2}\right)+(K-\mu_N)\left(\psi(b)-\psi\left(-\frac{\mu_N}{\sigma_N}\right)\right) \quad (5.47)$$

Putting the value of the equation (5.47) in equation (5.46) we get the price of a HDD put option as:

$$p(t)=e^{-r(t-t_0)}\left[\frac{\sigma_N}{\sqrt{2\pi}}\left(e^{-\frac{b2}{2}}-e^{-\frac{1}{2}\left(\frac{-\mu N}{\sigma N}\right)^2}\right)+(K-\mu_N)\left(\psi(b)-\psi\left(-\frac{\mu_N}{\sigma_N}\right)\right)\right] \quad (5.48)$$

In the above equations, Ψ is the cumulative distribution function for the standardized normal distribution. In the same procedure the option pricing for CDDn contracts can be formulated under out-of-period valuation framework.

5.2.3 In -Period Valuation

To calculate the price of an option for the time period (t), where time frame for the month is denoted as $[t_1,t_m]$ and $t_1 \le t \le t_m$, the model is altered. The out-of-period valuation takes the forecasted value of the temperature, whereas in case of in-period valuation the temperature is known. This is the basic difference in valuing options under 'Out-of-period Valuation' and 'In-period valuation'. Now, modifying equation (5.40) to get the in-period valuation for the heating-degree days call option we get:

$$c(t)=e^{-r(t-t_0)}E_Q\left[\sum_{i=i}^{t}\max\{(H_N-K),0\}\mid F_t\right] \quad (5.49)$$

Now, solving the right hand side of the above equation we get:

$$E_Q\left[\sum_{i=i}^{t}\max\{(H_N-K),0\}\mid F_t\right]=\sum_{i=1}^{t}E_Q\left[\max\{(H_N-K),0\}\mid F_t\right]$$

$$=\sum_{i=1}^{t}\left[\max\{(H_N-K),0\}\right]+\sum_{l=t+1}^{m}E_Q\left[\max\{(H_l-K),0\}\mid F_t\right] \quad (5.50)$$

Putting the values from equation (5.50) in equation (5.49) we get the in-period valuation of the HDD call option:

$$c(t)=e^{-r(t-t_0)}\left[\sum_{i=1}^{t}\left[\max\{(H_N-K),0\}\right]+\sum_{l=t+1}^{m}E_Q\left[\max\{(H_l-K),0\}\mid F_t\right]\right] \quad (5.51)$$

Similarly, the value of HDDn put option under in-period valuation framework is:

$$p(t)=e^{-r(t-t_0)}\left[\sum_{i=1}^{t}\left[\max\{(K-H_N),0\}\right]+\sum_{l=t+1}^{m}E_Q\left[\max\{(K-H_l),0\}\mid F_t\right]\right] \quad (5.52)$$

The first part of equation (5.50) is know as it is recorded at 't' and the second part of the equation considers the forecasted value of temperature for the time period 'l', day of the month, to the end of the month t_m. Hence it is stochastic in nature. In the same procedure the option pricing for CDDn contracts can be formulated.

However, since the reference temperature (18^0C) level is set under US market conditions and it is also accepted in Europe, the models represented by equations (5.45, 5.48, 5.51 and 5.52) are only bound to winter seasons. Under restrictions the above equations can be used for the summer-period valuations. The difficulty arises once the mean temperature exceeds the reference level, in such a scenario we get: ($\max\ [18 - T_i] \neq 0$). The pricing of these options during these months follows a process which can be estimated using Mote Carlo simulation techniques as explained in the later section.

5.2.4 Estimating the market price of risk (b)

The other parameter which has to be estimated in order to calculate the degree-days option prices is the risk premium factor (b). This parameter is taken as a constant in the model formulation for the option prices. This section of the study defines the approach to calibrate the market price of risk.

To provide a precise estimate of the of the risk premium estimator we minimize the following objective function:

$$\min \sum_{t=1}^{n} w_t \left(P_t - \hat{P}_t\right)^2 \qquad (5.53)$$

The P_t and the capped P_t values denote the recorded and calculated price of the CDD or HDD price for specific stations respectively. The objective function minimizes the squared difference value of the observed and calculated degree-days price values. The term is squared in order to achieve the true distance between the actual and the calculated values. Since, the market of weather derivatives is not as liquid compared to other financial markets, the squared difference term is weighted against its volume traded (V_t) in that particular day. Therefore, we define the weight as:

$$wt = \frac{Vi}{\sum_{i=1}^{m} \sum_{t=1}^{n} V_{it}} \qquad (5.54)$$

This study calibrates the model to the market by using the value of the risk premium factor (b) calculated by above procedure.

6. Results

6.1 Parameter Estimation

6.1.1 The Noise process

The parameter values of the variance of noise process estimated as per equation (5.16) for different months are shown below:

IV - Table 6.1: Estimated values of noise-variance

Months	London	Louisville	Nashville	Bristol
January	6.23	5.72	4.98	6.15
February	5.60	5.96	5.18	4.77
March	4.90	5.04	4.49	4.41
April	3.89	3.54	3.27	2.96
May	3.06	3.44	2.77	2.98
June	2.06	2.53	2.28	2.48
July	1.90	2.33	1.58	1.72
August	1.63	2.31	2.01	1.35
September	1.36	2.40	1.70	1.17
October	2.74	3.15	3.09	2.72
November	3.18	3.71	2.94	2.58
December	4.83	4.47	5.09	4.97

The results reveal that the noise-variances during the winter months are relatively higher than the summers.

6.1.2 Mean Reversion

The estimated value of mean reversion calculated based on equation (5.17) is shown below:

V - Table 6.2: Estimated values of Mean-reversion factor

Parameter	London	Louisville	Nashville	Bristol
⊝	0.395	0.389	0.391	0.392

6.2 The Temperature Process

The following table shows the estimated parameters of the temperature process modelled under equation (5.2) for the four cities. The parameters are estimated based on the equations (5.12 to 5.15).

VI - Table 6.3: Estimated values of Temperature Process parameters

Parameters	London	Louisville	Nashville	Bristol
B_o	6.78494	6.25982	8.16966	3.85403
B_1	0.11139	0.13067	0.11361	0.17486
B_2	-0.00032	-0.00036	-0.00032	-0.00045
a	16.24	19.26	16.71	18.60
∅	1.314	1.268	1.296	1.314
2πω	0.017	0.017	0.017	0.017

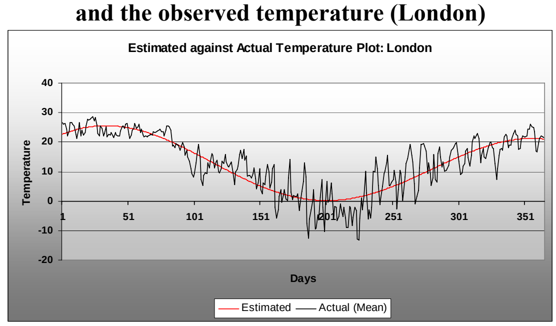
The mean temperature function based on the equation (5.2) from the time-series data set for London is:

$$\overline{T}_t = 6.78 + 0.11t - 0.0003t^2 + 16.24\sin\left(0.017t + 1.31\right) \qquad (6.1)$$

T-Value: (3.25) (3.45) (-3.61)

A 'α' value of 16.24^0C of London suggests that the difference between the mean temperature in a particular day during summer and winter months is 13.14^0C. The figure below shows the plot of the function along with the temperature data.

IV - Figure 6.1: The estimated modelled temperature and the observed temperature (London)



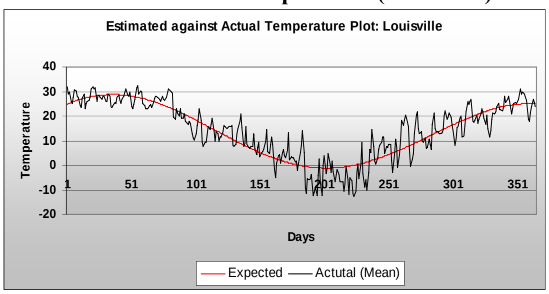
The mean temperature function based on the equation (5.2) for Louisville is:

$$\overline{T}_t = 6.25 + 0.13t - 0.00036t^2 + 19.26\sin(0.017t + 1.26) \quad (6.2)$$

T-Value: (2.82) (3.80) (-3.81)

Here in the above equation the 'α' value of 19.26^0C of Louisville suggests that the difference between the mean temperature in a particular day during summer and winter months is 12.6^0C. The figure below shows the plot of the function along with the temperature data.

V - Figure 6.2: The estimated modeled temperature and the observed temperature (Louisville)



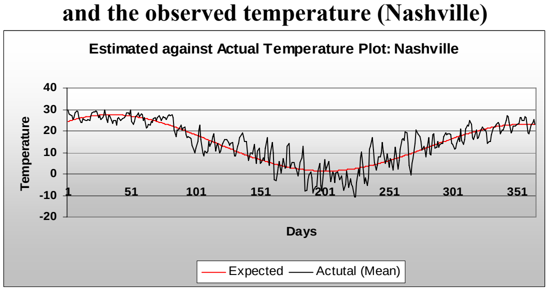
Similarly, the mean temperature function based on the equation (5.2) from the time-series data set for Nashville is:

$$\overline{T}_t = 8.17 + 0.11t - 0.0003t^2 + 16.71\sin(0.017t + 1.29) \quad (6.3)$$

T-Value: (4.18) (3.75) (-3.90)

Here the 'α' value of 16.71^0C of Nashville suggests that the difference between the mean temperature in a particular day during summer and winter months is 12.9^0C. The figure below shows the plot of the function along with the temperature data.

VI - Figure 6.3: The estimated modeled temperature and the observed temperature (Nashville)



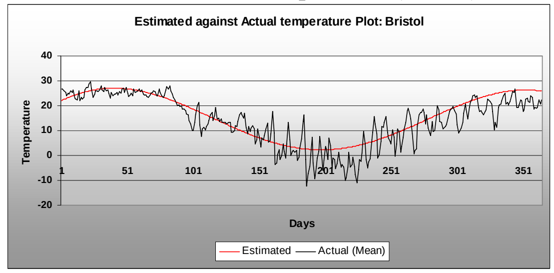
The mean temperature function based on the equation (5.2) for Bristol is formed as:

$$\overline{T}_t = 3.85 + 0.17t - 0.0004t^2 + 18.6\sin(0.017t + 1.34) \quad (6.4)$$

T-Value: (1.97) (5.75) (-5.89)

Similarly here a 'α' value of 18.6^0C of Bristol suggests that the difference between the mean temperature in a particular day during summer and winter months is 13.4^0C. The figure below shows the plot of the function along with the temperature data.

VII - Figure 6.4: The estimated modeled temperature and the observed temperature (Bristol)



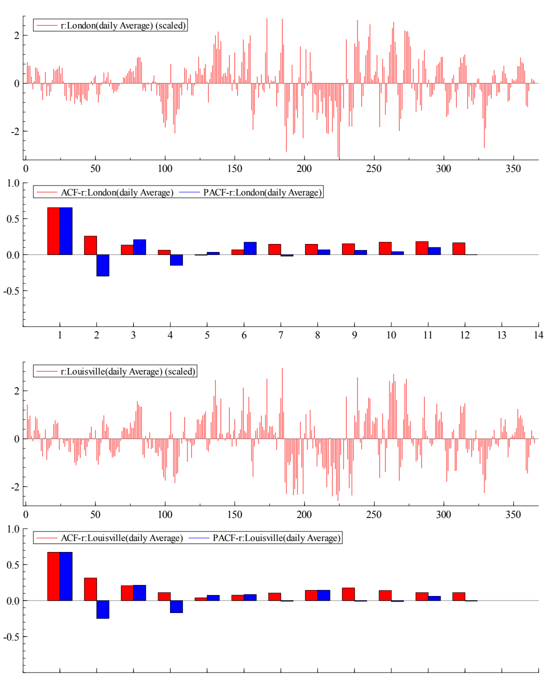
These graphs indicate a good fit of temperature data; the residuals are stationary with no serial correlation.

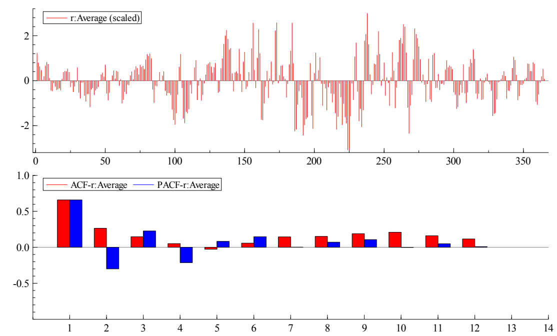
6.2.1 Volatility Pattern

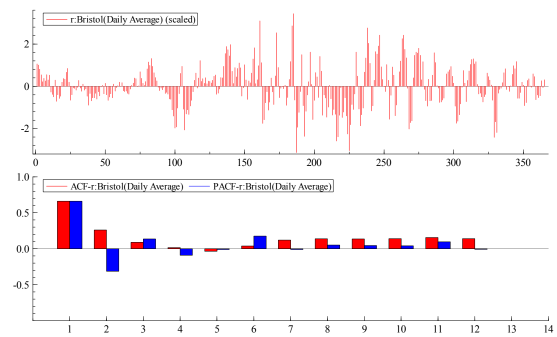
From the plot of the temperature function against mean temperature in figure (5.1) it is clear that there exits a non-uniformity of volatility over the years. The figure also reveals a prominent seasonal pattern in the temperature fluctuations. A quadratic seasonal trend is used in the temperature model to capture the long-term seasonal trend.

On removing the estimated parameters from the time-series temperature data series, we obtain the residuals which should be normally distributed provided the model assumptions are accurate. The figure below shows residual distribution of the temperature model.

VIII - Figure 6.5: Residual Distribution







The above figures of all the four cities suggest that the residuals are a proper estimate of a normal distributed and hence it assures that the model estimations are correct.

6.3 *Option Pricing*

6.3.1 Equating the model to the market

The following table shows risk premium factor of the various cities calibrated over the winter months.

VII - Table 6.4: Estimated value of Market Price of Risk

Parameter	London	Louisville	Nashville	Bristol
B	0.08	0.095	0.24	0.137

From equation (1) it is clear that the temperature model can be equated to the market as per the needs of the contract. If a higher temperature is expected during the time of the contract, the constant term in the temperature-model equation can be increased to obtain an adjacent decreased value in H_N, for the contract period. The values of variation (s) and the amplitude (a) of the pricing model and temperature equation can be altered to capture the meteorological data.

The following table reports the result of the prices calculated as per the derived option model, equation (5.45), for the HDD call option.

VIII - Table 6.5: Estimated Price of HDD Call option

Weather Station	London	Louisville	Nashville	Bristol
Index	HDD	HDD	HDD	HDD
Type	Call	Call	Call	Call
Period	January	January	January	January
Strike	600 HDDs	600 HDDs	600 HDDs	600 HDDs
Nominal	1GBP/HDD	1GBP/HDD	1GBP/HDD	1GBP/HDD
Option Price	23.92	27.14	50.04	42.21

The following table reports the result of the prices calculated as per the derived option model, equation (5.48), for the HDD put option.

IX - Table 6.6: Estimated Price of HDD Put option

Weather Station	London	Louisville	Nashville	Bristol
Index	HDD	HDD	HDD	HDD
Type	Put	Put	Put	Put
Period	January	January	January	January
Strike	600 HDDs	600 HDDs	600 HDDs	600 HDDs
Nominal	1GBP/HDD	1GBP/HDD	1GBP/HDD	1GBP/HDD
Option Price	19.90	23.32	46.39	37.91

The prices obtained here vary in accordance to the mean temperature of the weather station. It can be further observed that in spite of having the same strike level the option prices differ in accordance to the risk factor of the weather derivative relative to the particular weather station. Hence, it is of great importance to have an accurate value of the risk premium, which is estimated here by least square differences method.

6.4 *Monte Carlo Simulation*

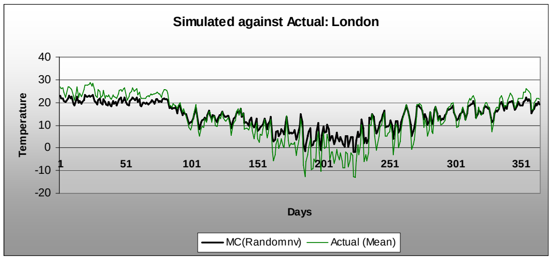
To perform a Monte Carlo simulation of the stochastic process equation (5.6) is approximated as an Euler function of the given form:

$$T_{+1t} - T_t = \frac{d\overline{T}_t}{dt} - \theta\left(T_t - \overline{T}_t\right) + \sigma_t Z \qquad (6.5)$$

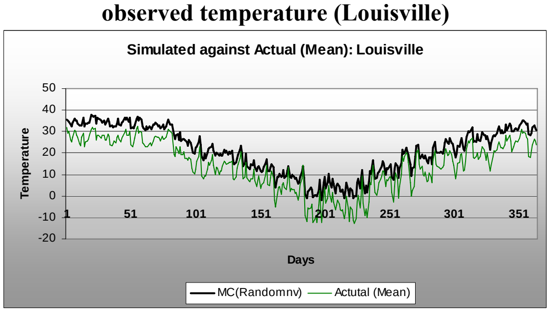
In the given equation we take: Z~N(0,1). The above equation is evaluated to simulate trajectories of the temperature stochastic process using a random number generator using 10000 sample paths.

The figure below shows the sample trajectories of the various stations considered to find the expected payoffs of the degree-days options against the observed daily mean temperature.

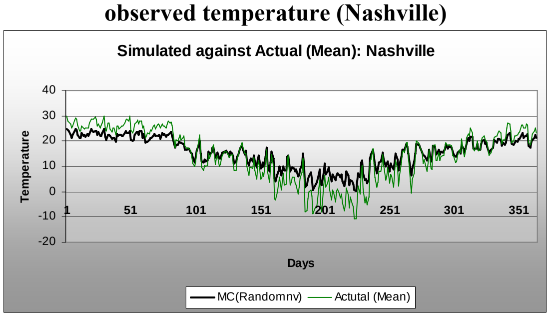
IX - Figure 6.6: The simulated temperature and the observed temperature (London)



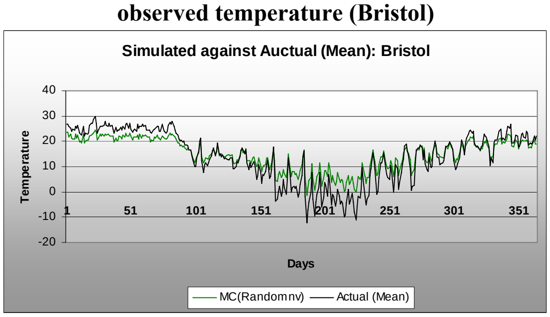
X - Figure 6.7: The simulated temperature and the observed temperature (Louisville)



XI - Figure 6.8: The simulated temperature and the observed temperature (Nashville)

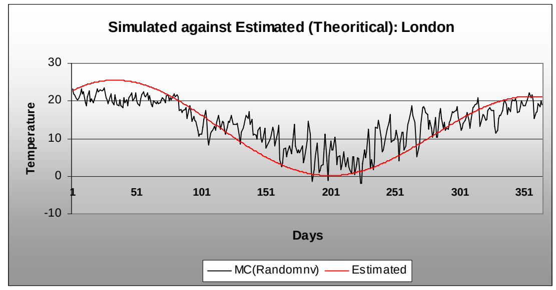


XII - Figure 6.9: The simulated temperature and the observed temperature (Bristol)

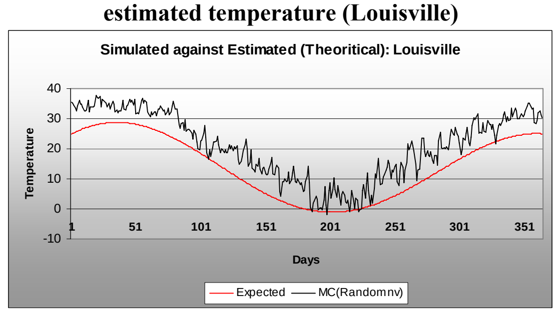


The plot of simulated temperature against calculated expected mean temperature is shown below.

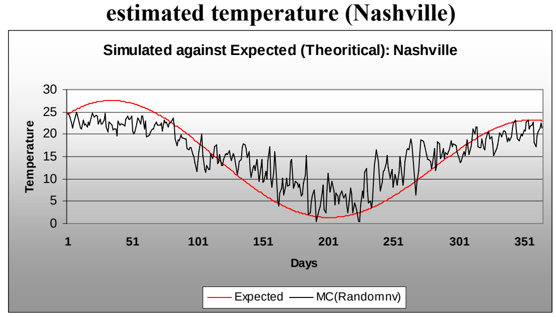
XIII - Figure 6.10: The simulated temperature and the estimated temperature (London)



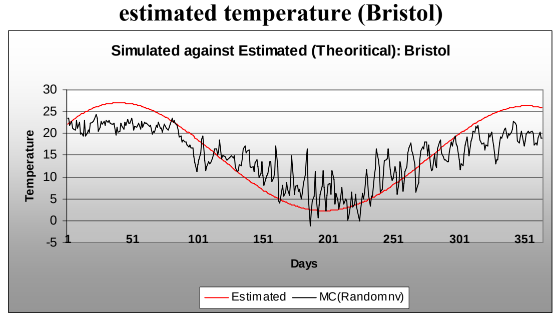
XIV - Figure 6.11: The simulated temperature and the estimated temperature (Louisville)



XV - Figure 6.12: The simulated temperature and the estimated temperature (Nashville)



XVI - Figure 6.13: The simulated temperature and the estimated temperature (Bristol)



7. Risk Management Applications

The weather derivatives have gained much importance in the contemporary world among various industry segments to hedge their weather risk faced due to unfavourable weather conditions. These instruments act as a primary tool for risk management purpose in different ways for various industries. The table below shows ways in which different industries are affected by adverse weather conditions.

X - Table 7.1: Weather variables which affect various industries

Industry	Weather variable
Construction	Temperature, snowfall
Agriculture	Temperature, precipitation, snowfall
Beverage	Temperature
Tourism	Temperature, precipitation, snowfall
Entertainment	Precipitation

The following two examples define the way in which beverage and construction industries are affected by these weather variables.

7.1 Beverage Industry

Garcia and Sturzenegger (2001) illustrated the importance of weather derivative as an instrument to hedge weather risk. In a case study carried out by them they highlighted the effect of temperature on milk and juice sales in four European cities, Switzerland, France, Germany and United Kingdom. The authors stated that a positive correlation appeared between the juice sales and the daily mean temperature. Garcia et al, further investigated the significance level of this positive correlation among the various countries and aimed to define a level of mean temperature beyond which the impact on sales was considerably reduced.

A chain of wine bars operator based in London, Corney and Barrow, investigated that their sales reduced significantly when there was a noticeable decrease in the daily mean temperature. In order to hedge their weather risk and reduce the effect of temperature on their sales return, they entered into degree-days contract in the year 2000, May. The contract had an initial disbursement of GBP 1500 on days the temperature dropped below 24^0C. The maximum payoff was capped to GBP 15000 per day and GBP 100000 for the entire summer season. The contract provided an efficient hedge against the weather risk and was only executed on Thursdays and Fridays which were affected the most due to temperature fluctuations.

7.2 Construction Industry

The construction industry is influenced by the daily temperature in a number of ways. This has been the primary reason for many investors from this industry segment to enter into weather derivatives contract. While increasing day temperature results in delay of work, cold temperature has a major negative impact on the cash flows of a company. The common hazards faced due to weather fluctuations by majority of the industry participants are:

I. Hindrance to work: Workers at the construction site are exposed to weather fluctuations during the day. Hot weather conditions reduce the efficiency of work, whereas extreme conditions can put work to a halt.

II. Concrete: Extreme cold temperature and frost in the climate causes the water to freeze in the concrete structures. These lead to high possibilities of crack formation in these structures. Low temperature can cause delay in the construction of these concrete structures.

III. Low Temperature and snow fall: An unfavourable extreme low temperature weather condition causes extreme delay in the working environment and can even be a reason for the temporary built structures to collapse.

8. Conclusion

The weather derivatives have witnessed much importance in the contemporary financial market as an instrument to protect firms from the exposure to unfavourable weather conditions, weather risk. The Chicago Mercantile Exchange has further fuelled the significance of the emerging weather derivatives market and this triggered the development of dynamic pricing model of degree-day derivatives to analyze their risk exposure. However, the weather derivatives market is at its evolving stage and has a long way to go when compared to other financial markets. This study deals with the development of an accurate model for the pricing of derivatives by precisely addressing the fluctuations of the instrument's underlying variable, temperature.

This dissertation has analyzed the daily temperature data for four UK cities namely London, Bristol, Louisville and Nashville for a period of thirty years, 1st July 1978 to 1st July 2008. Based on the statistical characteristics of the daily temperature variations, a Gaussian Ornstein-Uhlenbeck model has been equated considering time-variant mean and volatility to address the stochastic nature of temperature movements. In order to capture the seasonal dynamics, the model also considers a truncated Fourier series to represent its Harmonic-Phasors. Furthermore a quadratic time-dependent term is added to address the long-term upward trend in the temperature. Importantly, it should be taken into account that the seasonal fluctuations of the underlying variable of weather derivative, temperature, differs significantly across stations in both amplitude and frequency. The formulated model is refined enough capture the temperature trajectory of the locations and forecast an accurate temperature movement.

With regards to pricing of weather derivatives the dissertation provides formula for option prices based on heating degree-days (HDD) and cooling degree-days (CDD) indices. The study evaluates an arbitrage-free option pricing using a Gaussian Ornstein-Uhlenbeck model. The pricing equation is solved under a risk-neutral frame work defining a Q-martingale process to find the density function. This study calculates both out-off-period valuation and the in-period valuation of weather derivatives.

Finally, considering weather derivatives to be an incomplete market the risk premium factor plays a significant role in futures pricing of degree-days derivatives. This parameter is estimated by minimizing an objective function of the squared differences of the calculated prices and the observed quoted prices available in the market. This estimated market risk factor is significantly different from zero and tends to show a time-dependent pattern. However, there lies further scope of research in the field of pricing of weather derivatives with regards to the analysis of the above stated risk premium factor and its seasonal time-varying pattern which will enable to improve the pricing of HDD and CDD options. Importantly, the pricing equation may be further improved by developing the temperature model in a way to include a more refined model to define the driving noise process of the temperature variations and its seasonal pattern. Perhaps, a way to do this is to consider temperature as one of the factors of a larger model explaining the climatic variations.

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