The Fama and French Model and Leverage:

Compatibility with the Modigliani and Miller Propositions.

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Abstract

Fama and French (FF) argue that their 3-factor asset pricing model can be interpreted as representative of equilibrium pricing models in the spirit of Merton's (1973) inter-temporal CAPM or Ross's (1976) arbitrage pricing theory (APT) (FF, 1993, 1994, 1995, 1996). Such claims however are compromised by Lally's (2004) observation that loadings on the FF portfolio risk factors consistent with FF (1997) lead generally to outcomes that are contradictory with rational asset pricing in the context of leverage. In response, we outline a framework of asset pricing consistent with the Modigliani and Miller propositions in the context of leverage, within which framework it is possible to interpret the FF 3-factor model as a proxy representation. Our outcome approach to leverage leads by construction to compatibility of the FF 3-factor model with the Modigliani and Miller propositions of rational pricing.

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1. Introduction

The issue of whether the Fama and French (FF) 3-factor model is consistent with the Modigliani and Miller (MM) (1958, 1963) propositions has received surprisingly little attention (Lally (2004) appears as something of an exception). The MM propositions capture the principal that equations of risk and return that purport to reflect rationality in perfect capital markets (including no tax) should comply with a risk-reward structure for the firm's debt and equity holders in combination that does not change with variations of leverage. The intuition is of no-arbitrage: "the firm is the firm": whose cash return to investors derives from its output goods and services, which need not be affected by financial leverage.

If the FF-model is to be viewed as the outcome of systematic mispricing – the outcome of behavioural or psychological propensities (for example, Lakonishok, Shleifer and Vishny 1994; Daniel, Hirshleifer and Subrahmanyam, 2001) - we have little basis for expecting that the model's outcome implications should adhere to a systematic rational framework as the Modigliani and Miller propositions. We would, in effect, be obliged to move to an empirically-driven edifice, whereby one empirical observation has equal standing with another. A theoretical or rational development of a model that of itself flies in the face of theoretical rationality is always likely to falter against the next empirical observation. As Lally points out, the "price" of diverging from arbitrage models, such as those of Merton (1973) and Ross (1976), is that the "logical" developments that are the outcome of the Modigliani and Miller propositions (the equations for the firm's cost of equity, its *WACC* and beta as a function of leverage, for example) are made redundant.

Fama and French (1993, 1995, 1996, 1997) have aggressively interpreted their model as consistent with Merton's (1973) inter-temporal CAPM (ICAPM) and Ross's (1976) arbitrage pricing theory (APT). In this view, the big-minus-small (*BMS*) "market size" portfolio and high-minus-low (*HML*) "book-to-market equity" portfolio factors in their model are viewed as mimicing underlying risk factors that can be hedged by investors. They note that the co-variabilities of return for the stocks of such firms do not appear to be captured by the on-going movements of the market's returns, but do appear nevertheless to be compensated for in average returns (Huberman and Kandel, 1987; Chan and Chen, 1991; FF, 1995). So the *BMS* and *HML* portfolios are interpreted as capturing premiums for risk as it affects the cash flows of firms, the outcome of both "more distressed" firms (with higher book-to-market equity ratio) and "less-responsive-to-economic-cycles" (smaller) firms having less reliable earnings. Hence the requirement for asset loadings on risk factors in addition to the market return, to reflect such aspects of risk exposure.

If the Fama and French 3-factor model represents a version of Merton's ICAPM or Ross's APT, it clearly must conform with the fundamental principals of arbitrage as captured by the Modigliani and Miller (MM) propositions. On the face of it, however, the FF factor model is algebraically inconsistent with the MM propositions and hence with the no-arbitrage conditions of leverage. Lally (2004) demonstrates strikingly how by applying an increased leverage to a firm, the FF 3-factor model can quite easily be made to predict that leverage of itself can cause the cost of equity *and* the *WACC* to simultaneously fall (which contradicts all notions of mean-variance theory: that higher equity risk is commensurate with a higher cost of equity capital). In one industry, Lally actually finds that the *WACC* declines with leverage to the point of being negative. In such manner, Lally observes that substantial violations of rationality are inherent in the Fama and French (1997) expressions for the loadings on the risk factors (*BMS* and *HML*). To attain the required consistency with the MM propositions, Lally applies a "leverage patch" to the FF (1997) loadings on the market size and book-to-market equity factors. Notwithstanding, Lally concedes that there is little empirical justification for his reconstructions

The present paper considers the FF 3-factor model from the perspective of the MM propositions. To this end, we proceed from the direction of demonstrating a broad class of multi-factor asset pricing models that are compatible with the MM propositions. We then turn to consider whether it is reasonable to consider the FF 3-factor model as a proxy for such a model. In principle, we conclude that it is possible to do so. In line with such a perspective, we present our preferred approach to leverage that maintains consistency between the FF 3-factor model and the MM propositions.

The rest of the paper is arranged as follows. In the following section, we outline the framework of a general multi-factor asset pricing model that is compatible with the MM propositions. In the section thereafter, we consider that Fama and French's (1997) empirically-derived expressions for loading on the portfolio factors are not algebraically consistent with leverage. Notwithstanding, the following section examines the case for the FF 3-factor model as a proxy for a rational risk-return model of asset prices. The penultimate section presents our recommended approach to leverage of the FF loadings that allows for compatibility of the FF 3-factor model with the MM propositions. The final section concludes the paper.

2. A General Asset Pricing Model Consistent with the MM Propositions

In this section, in the spirit of arbitrage pricing theory (APT), we develop a general asset pricing model (of which the CAPM is a special case) consistent with the Modigliani and Miller propositions.

In the absence of market frictions including corporate tax, Modigliani and Miller demonstrate that the overall return on the levered firm (R_F) (debt plus equity) - which is the weighted average of the returns on the firm's equity (R_E) and its bonds (R_B) – must be equal to the return on the otherwise-equivalent unlevered firm (equity only) (R_U). That is $R_F = R_U$:

$$R_F \equiv \frac{B}{B+E} R_B + \frac{E}{B+E} R_E = R_U \tag{1}$$

The intuition is that the firm's cash returns to its debt and equity holders derive from the firm's operations, which are assumed to be independent of the firm's financing arrangements. It follows that the market value of the levered firm (V_F) – equity (V_E) plus bonds (V_B) – is that of the unlevered counterpart firm (V_U):

$$V_F = V_E + V_B = V_U. \tag{2}$$

which is Modigliani and Miller's proposition I. A violation of the above implies an opportunity to arbitrage. Allowing corporate tax (at rate T_c) and assuming that the debt is held in perpetuity, the additional after-tax cash flow to the firm created by the tax-deductibility of the interest payments may be expressed as $B.R_B.T_c$ in each period, where B is the market value of the firm's debt with market interest rate R_B. Assuming additionally that such cash flows have the same risk as the debt itself, and summing the flows as a perpetuity, their market value is $B.T_c.R_B / R_B = B.T_c$. In which case, we have:

$$V_F = V_E + V_B = V_U + B.T_c \tag{3}$$

It follows that the additional "tax shield" value for the levered firm ($B.T_c$) has an expectation of return equal to that of the underlying bonds (R_B). Hence allowing corporate tax, we have:

$$R_F \equiv \frac{B}{B+E} R_B + \frac{E}{B+E} R_E = \frac{B+E-BT_c}{B+E} R_U \;+\; \frac{BT_c}{B+E} R_B$$

yielding:

$$R_E = \left[1 + \frac{B}{E}(1-T_c)\right] R_U - \frac{B}{E}(1-T_c) R_B \qquad (4)$$

which is Modigliani and Miller's proposition II.[1] Defining the weighted average cost of capital (*WACC*) as:

$$WACC_E = \frac{E}{V} R_E + \frac{B}{V}(1-T_c) R_B \qquad (5)$$

with V defined as the sum of E and B and substituting in equation 5 with R_E from equation 4, gives:

$$WACC_E = R_U \left[1 - \frac{B}{V} T_c\right] \qquad (6)$$

which is Modiglaini and Miller's proposition III.

Assuming no corporate tax, the overall beta for the levered firm (β_F) (debt plus equity) – which, by the mathematics of covariance, is the weighted average of the betas on the firm's equity stock (β_E) and bonds (β_B) – is equal to the beta of the unlevered firm (equity only) (β_U). That is $\beta_F = \beta_U$:

$$\beta_F = \frac{E}{B+E} \beta_E + \frac{B}{B+E} \beta_B = \beta_U$$

(7)

With corporate tax, the "additional" value for the levered firm ($B.T_c$) has beta equal to that of the underlying bonds (β_B). In this case we have:

$$\beta_F = \frac{E}{B+E}\beta_E + \frac{B}{B+E}\beta_B = \frac{B+E-BT_c}{B+E}\beta_U \ + \ \frac{BT_c}{B+E}\beta_B$$

giving:

$$\beta_E = \left[1+\frac{B}{E}(1-T_c)\right]\beta_U - \frac{B}{E}(1-T_c)\beta_B \qquad (8)$$

Combining equations 4 and 8, we have:

$$\frac{R_E - R_U}{R_U - R_B} \ = \ \frac{\beta_E - \beta_U}{\beta_U - \beta_B} \qquad (9)$$

In the case that the debt has a risk-free return R_f ($\beta_B = 0$) and the unlevered firm has a return equal to the return on the market (R_m) ($\beta_m = 1$), equation 9 gives:

$$R_E - R_f \ = \ \beta_E.(\ R_m - \ R_f) \qquad (10)$$

as a mathematical consequence of leverage. Taking expectations:

$$E(R_E) \ - \ R_f \ = \ \ \beta_E.(E(R_m) \ - \ R_f) \qquad (11)$$

which is the CAPM. The CAPM is thereby demonstrated as consistent with the leverage equations 4 and 8 and hence with Modigliani and Miller's propositions I, II and III. More generally, we can show that a model for the return R_j on the general firm j of the form:

$$R_j \ - R_f = \ b_j.(R_m - R_f) \ + s_j.R_1 \ + \ h_j.R_2 + \ . \ . \ . \qquad (12)$$

where the R_m, R_1, R_2 . . . are risky variables and the b_j, s_j, h_j . . . are either held constant with leverage or with leverage act as:

$$b_j \ = \left\{\left[1+\frac{B}{E}(1-T_c)\right]b_{U,j} - \frac{B}{E}(1-T_c)b_{B,j}\right\} \qquad (13)$$

$$s_j \ = \left\{\left[1+\frac{B}{E}(1-T_c)\right]s_{U,j} - \frac{B}{E}(1-T_c)s_{B,j}\right\} \qquad (14)$$

$$h_j \ = \left\{\left[1+\frac{B}{E}(1-T_c)\right]h_{U,j} - \frac{B}{E}(1-T_c)h_{B,j}\right\} \qquad (15)$$

where $b_{U,j}$, $s_{U,j}$ and $h_{U,j}$ are the values of b_j, s_j, h_j for the unlevered firm, respectively, and $b_{B,j}$, $s_{B,j}$ and $h_{B,j}$ are the values of b_j, s_j, h_j for the firm's debt, respectively, is equally consistent with the Modigliani and Miller propositions. To see this, observe that equation 12 implies:

$$b_j = (R_j - R_f - s_j.R_1 - h_j.R_2 - \ldots) / (R_m - R_f)$$

$$b_{U,j} = (R_{U,j} - R_f - s_{U,j}.R_1 - h_{U,j}.R_2 - \ldots) / (R_m - R_f) \quad (16)$$

$$b_{B,j} = (R_{B,j} - R_f - s_{B,j}.R_1 - h_{B,j}.R_2 - \ldots) / (R_m - R_f)$$

Substituting equations 16 into equation 13 (with equations 14 and 15) gives:

$$R_j = \left[1 + \frac{B}{E}(1 - T_c)\right] R_{U,j} - \frac{B}{E}(1 - T_c) R_{B,j} \quad (17)$$

which is Modigliani and Miller's proposition II (*cf* equation 4). In perfect markets with no corporate tax, equations 13 – 15 are the statements that the total firm's exposure to the risky returns R_m, R_1, R_2. . . is neither created nor destroyed by variations of leverage. In effect, such models state that an asset's expected return (R_j) relates to fixed variables and exposure to risky variables (R_m, R_1, R_2 . . ., as in equation 12) and that such exposures in perfect markets in the absence of corporate taxes are neither created nor destroyed for the firm as a whole by the firm's choice of capital structure. Simply by taking expectations of equation 12, we have the outcome that the expected return $E(R_j)$ for the general firm *j* may be expressed:

$$E(R_j) - R_f = b_j.[E(R_m) - R_f] + s_j.E(R_1) + h_j.E(R_2) + \ldots \quad (18)$$

is consistent with Modigliani and Miller's propositions, where the R_m, R_1, R_2 . . . are risky variables and the b_j, s_j, h_j . . . are either held constant with leverage or with leverage act as equations 13 - 15.

3. The Fama and French 3-Factor Model and Leverage

The Fama and French expectations model for the expected return $E(R_j)$ on a portfolio is expressed:

$$E(R_j) - R_f = b_j.[E(R_m) - R_f] + s_j.E(R_{SMB}) + h_j.E(R_{HML}) \quad (19)$$

with its implied unlevered counterpart:

$$E(R_{U,j}) - R_f = b_{U,j}.[E(R_m) - R_f] + s_{U,j}.E(R_{SMB}) + h_{U,j}.E(R_{HML}) \quad (20)$$

where R_{SMB} is the expected rate of return on a portfolio of stocks of small firms less that on a portfolio of stocks of big firms; R_{HML} is the expected rate of return on a portfolio of stocks with high book-to-market equity less that on a portfolio of stocks with low book-to-market equity; and the coefficients b, s, and h are the "covariabilities" of the firm's equity returns with R_m, R_{SMB}, and R_{HML}, respectively. Fama and French (1996, 1997) state that using loadings on the *SMB* factor to explain stockmarket returns is in line with the evidence of Huberman and Kandel (1987) that there is co-variation in the returns of stocks of small firms that is not captured by the on-going or continuous market return, but is nevertheless compensated by average returns; and that using loadings on the *HML* factor to explain stockmarket returns is in line with the evidence of Chan and Chen (1991) that there is return co-variation related to relative distress that is missed by the market return, but is compensated in average returns. Thus Fama and French interpret their model as portraying a security's expectation of return dependent on the sensitivity of its returns to the market return together with the sensitivity of its return to the *SMB* and *HML* portfolios as mimicing additional risk factors.

Identifying equation 19 with equation 18, we have the result that the Fama and French model is consistent with the Modigliani and Miller propositions provided either that:

(1) the coefficients b_j, s_j and h_j are held constant with leverage, or

(2) the coefficients b_j, s_j and h_j with leverage act as equations 13-15.

In the particular case that the firm's bonds can be assumed to be risk-free, the weightings $b_{B,j}$, $s_{B,j}$ and $h_{B,j}$ on the debt in equations 13-15 can be assumed equal to zero, and the equations become:

$$b_j = \left[1+\frac{B}{E}(1-T_c)\right]b_{U,j}$$

$$s_j = \left[1+\frac{B}{E}(1-T_c)\right]s_{U,j} \tag{21}$$

$$h_j = \left[1+\frac{B}{E}(1-T_c)\right]h_{U,j}$$

as Lally (2004). Fama and French (1997) model s_j as a function of the firm's market value of equity (*ME*):

$$s_j = s_{1,j} + s_{2,j} \ln (ME_j) \tag{22a}$$

and h_j as a function of the firm's book equity to market equity:

$$h_j = h_{1,j} + h_{2,j} \ln (ME_j / BE_j) \tag{22b}$$

In this case, consistency with the Modigliani and Miller propositions requires either that s_j and h_j are held constant with leverage, or with leverage act as:

$$s_j = \left[1+\frac{B}{E}(1-T_c)\right].\left\{s_{U1,j} + s_{U2,j} \ln(ME_U)_j\right\} - \frac{B}{E}(1-T_c).s_{B,j}$$

$$h_j = \left[1+\frac{B}{E}(1-T_c)\right].\left\{h_{U1,j} + h_{U2,j} \ln\left(\frac{BE_U}{ME_U}\right)_j\right\} - \frac{B}{E}(1-T_c).h_{B,j} \tag{23}$$

where the formulations in {} are invariant to pure leverage changes. In general terms, it is clear that we can argue for neither s_j nor h_j as defined by equations 22 as either constant or determined as equations 23 with leverage. In the case that the firm enjoys risk-free debt ($s_{B,j} = h_{B,j} = 0$) equations 23 become:

$$s_j = \left[1 + \frac{B}{E}(1 - T_c)\right].\left\{s_{U1,j} + s_{U2,j} \ln(ME_U)_j\right\}$$

$$h_j = \left[1 + \frac{B}{E}(1 - T_c)\right].\left\{h_{U1,j} + h_{U2,j} \ln\left(\frac{BE_U}{ME_U}\right)_j\right\} \qquad (24)$$

as Lally (2004). In the case that the firm has risk-free debt, Lally demonstrates that the outcomes from applying equations 22 are inconsistent with equations 24.

It appears that we are obliged to conclude that the Fama and French 3-factor model (equations 19 and 20) is generally incompatible with the Modigliani and Miller propositions. The further implication is that the Fama and French 3-factor model cannot be accepted *analytically* (in an algebraic no-arbitrage sense) as a model of returns on the basis of risk exposures. It may, nevertheless, be that the Fama and French "market size" and "book-to-market equity ratio" portfolios are somehow representations or proxies for actual risk factors, and that the model is a representative or proxy model for an underlying true analytical description of stock returns in the spirit of their claims. We substantiate this possibility in the following section.

4. An Asset Pricing Model that is Consistent with the Leverage Equations: The Case for the Fama and French 3-Factor Model

Allowing that the Fama and French factors are to be interpreted as factors for "risk," such risk must in some sense relate to "return variability." For the market investor, such return variability may be classified logically as one of the following: (1) a more or less continuous or on-going variability of the market (modelled, for example, as

Brownian motion); (2) the possibility of a "non-continuous" variability that is *in addition* to the above and which may be modelled as the outcome of a "surprise" market correction or "shock" to the economic market; (3) a more or less continuous idiosyncratic or non-systematic variability of individual asset returns as held by the investor (which is held to be not diversified); and (4) the possibility of a non-systematic or idiosyncratic surprise or shock to the performance of individual assets as held by the investor (which is held to be not diversified). The required return on the market $E(R_m)$ may then be modelled logically as a "risk-free" rate plus the additional returns that are required to compensate for bearing risks. Representing the premiums for bearing the above risks (continuous market variability, shock market variability, continuous idiosyncratic variability, shock idiosyncratic variability) as respectively, $E(R_{mc})$, $E(R_{ms})$, $E(R_{ic})$, $E(R_{is})$ and assuming independence of these returns, we may represent the required return on the market in terms of its components as:

$$E(R_m) - R_f = E(R_{mc}) + E(R_{ms}) + E(R_{ic}) + E(R_{is}) \quad (25)$$

In the case that idiosyncratic components of risk are not rewarded, $E(R_{ic}) = E(R_{is}) = 0$, and equation 25 becomes:

$$E(R_m) - R_f = E(R_{mc}) + E(R_{ms})$$

If we allow that asset *j*'s co-variance of return with the market return (as typically measured with daily, weekly or monthly data) might differ significantly from asset *j*'s covariance with the market in the event of a "shock" to the market, we can express the excess return for asset *j* as a loading on the rewards for bearing "continuous" and "shock" risk as:

$$E(R_j) - R_f = b_{c,j}.[E(R_m) - E(R_{ms}) - R_f] + b_{s,j}.[E(R_m) - E(R_{mc}) - R_f] \quad (26)$$

where $b_{c,j}$ and $b_{s,j}$ represents the loadings for asset j on the continuous and shock components of the excess market return.

Allowing that the compensation returns for holding non-market risk, $E(R_{ic})$ and $E(R_{is})$, might not actually be zero, equation 26 becomes:

$$E(R_j) - R_f =$$

$$b_{c,j}.[E(R_m - R_{ms} - R_{ic} - R_{is}) - R_f] + b_{s,j}.[E(R_m - R_{mc} - R_{ic} - R_{is}) - R_f] + E(R_{ic,j}) + E(R_{is,j}) \quad (27)$$

At this stage, we can simplify greatly if we choose not to distinguish between market and non-market "shocks." That is, to allow that a component of market return $E(R_S)$ is for bearing both types of shocks combined. Without confusion, we can then identify $E(R_I)$ as the component of return for bearing idiosyncratic risk and write:

$$E(R_j) - R_f = b_{c,j}.[E(R_m - E(R_I) - E(R_S) - R_f] + E(R_{I,j}) + E(R_{S,j}) \quad (28)$$

where $E(R_{I,j})$ and $E(R_{S,j})$ are the return components of asset j for bearing idiosyncratic and shock volatility, respectively. It is not clear-cut analytically how the returns $E(R_{I,j})$ and $E(R_{S,j})$ for the j'th asset should relate to the market-averaged return elements $E(R_I)$ and $E(R_S)$. We might however hypothesise that $E(R_{I,j})$ and $E(R_{S,j})$ can be expressed as loadings on the return factors $E(R_I)$ and $E(R_S)$, respectively, and that the additional premium return for bearing either idiosyncratic or shock volatility is linear with volatility, so that:

$$E(R_j) - R_f = b_{c,j}.[E(R_m) - E(R_I) - E(R_S) - R_f] + \frac{IV_j}{IV}E(R_I) + \frac{SV_j}{SV}E(R_S) \quad (29)$$

where IV represents the market value-weighted average of the idiosyncratic volatilities of the individual market assets (with reward premium, $E(R_I)$); and SV represents the market value-weighted average of the shock volatilities of the individual market assets (with reward premium, $E(R_S)$); and IV_j is the idiosyncratic volatility of asset j and SV_j is the shock induced volatility of asset j. Equation 29 represents our hypothesised model of asset return expectations. In perfect capital markets, but allowing corporate tax, the variability of the firm's returns can be assumed dependent on leverage as:

$$IV_j = \left\{ \left[1 + \frac{B}{E}(1 - T_c) \right] IV_{u,j} - \frac{B}{E}(1 - T_c) IV_{B,j} \right\}$$

and (30)

$$SV_j = \left\{ \left[1 + \frac{B}{E}(1 - T_c) \right] SV_{u,j} - \frac{B}{E}(1 - T_c) SV_{B,j} \right\}$$

Hence equation 29 is compatible with the Modigliani and Miller propositions.

The question of how to test equation 29 poses problems since we know neither the return components $E(R_I)$ and $E(R_S)$ nor the idiosyncratic or shock volatilities (either for the market (IV and SV) or for the asset (IV_j and SV_j)). One reasonable approach might be to identify a firm's sensitivity to continuous idiosyncratic volatility as inversely proportional to the firm's market size variable (on the basis that the earnings of small firms are likely to be less stable) and consider the component of market return commensurate with such idiosyncratic volatility $E(R_I)$ as proportional to the expected rate of return on a portfolio of stocks of small firms less that on a portfolio of stocks of big firms, R_{SMB}. And, similarly, to identify a firm's sensitivity to shock volatility as proportional to the firm's book-to-market equity ratio (on the basis that the earnings of firms with suppresses equity values (higher book-to-market equity ratios) are likely to be more susceptible to economic shocks) and consider the component of market return commensurate with such shock volatility $E(R_S)$ as proportional to the expected rate of return on a portfolio of stocks with high book-to-market equity less that on a portfolio of stocks with low book-to-market equity, R_{HML}. The above discussion suggests the model:

$$E(R_j) - R_f = b_j.[E(R_m) - E(R_I) - E(R_S) - R_f] + s_j. E(R_{SMB}) + h_j. E(R_{HML}) \quad (31)$$

with its regression model equivalent:

$$R_j - R_f = \alpha_j + b_j.[R_m - R_I - R_S - R_f] + s_j. R_{SMB} + h_j. R_{HML} + \varepsilon_j \quad (32)$$

as a proxy model for the true model (equation 29) where the requirement of conformity with the MM propositions is that s_j and h_j behave with leverage as equations 24.

The difficulty is that the R_I and R_S in equation 32 are unknown. It is therefore always going to be difficult to perform the regressions to determine the b_j's as represented by equation 32. Probably the best we can do is allow the b_j's to act as loadings on the full excess market return $[R_m - R_f]$. In which case the best we can offer as a testable equation for the preferred model of equation 29 becomes:

$$R_j - R_f = \alpha_j + b_j.[R_m - R_f] + s_j. R_{SMB} + h_j. R_{HML} + \varepsilon_j \quad (33)$$

which is the market model ex-post for the FF-3-factor model of equation 19.

In the following section, we show how the issue of leverage can be handled effectively under the assumptions that the Fama and French model is a true risk-return model driven by investor rationality and the principle of no arbitrage.

5. A Recommended Approach to Leverage in the Fama and French 3-Factor Model

A particular feature of the Fama and French model as an approximation of a true fundamental analytical model (perhaps as equation 26) it that the model does not appear to handle leverage (as demonstrated by Lally, 2004). One approach is to replace Fama and French's equations 22 and 23 (above) for the loadings on s and h with our equations 24. This is the approach recommended by Lally (2004). This approach, however, is beset by difficulties. To begin, we have the difficulty of computation of the *unlevered* $s_{U1,j}$, $s_{U2,j}$, $h_{U1,j}$, $h_{U2,j}$ in the equations. Additionally, the assumption that the firm enjoys risk-free debt as assumed by Lally appears unreasonable in the context of distressed firms. So we must actually hold with our equations 24 – with the additional complication that we now require to estimate the s and h loadings for the firm's risky debt. Finally as observed by Lally, the approach has no empirical justification.

An alternative approach, and the one recommended here, is to take the Fama and French (1997) estimates for the $s_{1,j}$, $s_{2,j}$, $h_{1,j}$, $h_{2,j}$ across industries as representing an *averaging over leverage* for that particular industry. In other words, to assume that the outcome for any particular firm using their tables of recommended $s_{1,j}$, $s_{2,j}$, $h_{1,j}$, $h_{2,j}$ values relate to a firm of typical industry leverage. This assumption is in the spirit of the way the results are estimated by Fama and French. The Lally-type calculation whereby the Fama and French equations 22 are applied directly to a firm with an arbitrary level of leverage is then inappropriate. Rather, the approach should be to apply equations 22 to the firm of interest assuming that it has a typical industry leverage, and then re-lever with MM proposition II (equation 17) to compensate for the firm's particular leverage. [2]

As an example, we consider as Lally, a firm in the Beer industry (say, BeerZZ) with market equity \$4.6b, leverage debt/equity (B/S) = 0.1 (hence debt, B = \$0.46b), and a book-to-equity ratio = 0.51 (hence book value of equity = \$2.35 billion). Also as Lally, we assume for simplicity that there are no taxes and that the firm's total value is invariant to pure leverages. Here we assume a typical leverage for the Beer industry as 30% as reflects the typical US firm. For the Beer industry (as Lally) we take from Fama and French (1997) $b = 0.90$, $s_1 = 0.1$, $s_2 = -0.15$, $h_1 = 0.27$, $h_2 = 0.73$; and the market risk premiums for R_m-R_f, SMB, and HML, respectively, 0.052, 0.032, 0.054.[3] Also for simplicity as Lally, we assume that the firm is able to borrow at risk-free debt at 6%.

The process for application of leverage to the Fama and French 3-factor model would then follow three stages:

Stage 1: Re-adjust the leverage for the firm being considered (BeerZZ) to reflect the typical leverage for the industry. Thus, in this case, we require that BeerZZ borrow \$X billion such that:

$$\frac{0.46 + X}{4.6 - X} = 0.3$$

Hence X = \$0.71, and the new value of market equity falls to \$(4.6 – 0.71) = \$3.89 billion, and the book equity to \$(2.35 – 0.71) = \$1.64 billion, implying that *BE/ME* falls to 1.64/3.89 = 0.42. Substituting these values into equations 22 and 23 yields

$$s = 0.10 - 0.15 \ln(3.89) = -0.104; \text{ and } h = 0.27 + 0.73 \ln(0.42) = -0.36$$

and then into the Fama and French model equation 19:

$$R = 0.06 + 0.9(0.052) - 0.104\,(0.032) - 0.36(0.054) = 0.84$$

gives the cost of equity for the re-financed (typical industry leverage) BeerZZ as 8.4%.

Stage 2: Apply the Modigliani and Miller's proposition II (equation 17) to calculate the cost of equity for the unlevered version of BeerZZ. Here for convenience we have assumed as Lally that the debt is risk-free. Hence we have equation 17 as:

$$R_E = \left[1 + \frac{B}{E}\right] R_U - \frac{B}{E} R_B \qquad (36)$$

giving: $0.084 = [1 + 0.3\,] R_U - 0.3(0.06) = 0.078$

Hence the cost of equity for unlevered BeerZZ is 7.8%.

Stage 3: Apply the Modigliani and Miller's proposition II to calculate the cost of equity for the required levered versions of BeerZZ. Hence with leverage 0.10 as Lally's first example, we would calculate (equation 36):

$$R_E = [1 + 0.1]\ 0.078 \ - \ 0.1(0.06) \ = \ 0.08$$

And with leverage 0.7 as Lally's second example, we would similarly calculate:

$$R_E = [1 + 0.7]\ 0.078 \ - \ 0.7(0.06) \ = \ 0.127$$

Hence the cost of equity for the required levered versions of BeerZZ are 8% (10% leverage) and 12.7% (70% leverage).

The above approach allows us to accept the industry estimates of Fama and French (1997) and simultaneously, by construction, have the outcome returns remain consistent with Modigliani and Miller's proposition II.

6. Conclusion

The paper has served to clarify the conditions of leverage that must apply to a factor model if it is be considered as algebraically consistent with the rules of no arbitrage. The structure of the FF 3-factor model does not of itself reveal accordance with such conditions. It is possible however to interpret the FF model as a proxy for a true analytical model that does accord with the conditions but which is difficult to test. Under such conditions, the paper shows how leverage of the FF 3-factor model can be applied.

Notes.

1. Rather than assuming that the *market value* of the firm's debt is held fixed in perpetuity, it may be more realistic to assume that the firm's *leverage ratio* (B/E) is held fixed in perpetuity, with the implication that the firm's corporate tax shields have the same risk as the unlevered firm (see, for example, Taggart, 1991). Modigliani and Miller's proposition II (equation 4 of the text) then becomes:

$$R_E = \left[1 + \frac{B}{E}\right] R_U - \frac{B}{E} R_B$$

and the corresponding relationship between levered and unlevered betas (equation 8 of the text) becomes:

$$\beta_E = \left[1 + \frac{B}{E}\right] \beta_U - \frac{B}{E} \beta_B$$

It follows that equations 13-15 and 17 (section 2), equations 21, 23 and 24 (section 3) and equation 30 (section 4) can be derived alternatively with the (1 - T_c) term everywhere eliminated. However, to comply with the approach of Lally (2004) we have allowed for the "intrusion" of the (1-T_c) term in our development of sections 3 and 4.

2. This is the approach as we would recommend if manipulating industry betas with leverage: namely, in the case that the leverage for BeerZZ is not typical of the industry, do *not* assume that the beta for BeerZZ is as for the Beer industry. Rather,

assume that the beta provided for the Beer industry is for a firm which has typical leverage in that industry. Then proceed to de-lever to find the beta for the unlevered firm in the Beer industry, before re-levering with the leverage of BeerZZ to determine its beta.

3. Fama and French do not show how the *b* coefficients on the market premium R_m - R_f might be adjusted with leverage. Accordingly, Lally (2004) in his example assumes that *b* (= 0.90) is invariant with leverage.

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